# Online Appendix
# Determinacy and E-stability with Interest Rate Rules at the Zero Lower Bound*


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## APPENDIX A: EQUATIONS

For convenience, we reproduce the following useful equations from the manuscript:

$$x_t = A(s_t)E_t x_{t+1} + B(s_t)x_{t-1} + C(s_t) + D(s_t)v_t, \tag{A1}$$

$$x_t = \Omega(s_t)x_{t-1} + Q(s_t)v_t + \Gamma(s_t), \tag{A2}$$

$$x_t = a(s_t) + b(s_t)x_{t-1} + c(s_t)v_t + \tilde{\epsilon}_t. \tag{A3}$$

Equation (A1) appears as equation (9) in the manuscript and describes the economic model. Equation (A2) appears as equation (10) in the manuscript and describes MSV REE. Equation (A3) appears as equation (11) in the manuscript and describes the adaptive learning agents' perceived law of motion (PLM).


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# APPENDIX B: PROOFS

## *B.1. Proof of Proposition 1*

Consider a slightly modified version of the IT Taylor rule:

$$i_t = \tilde{s}_t(\phi_\pi \pi_t + \phi_y y_t) - (1 - \tilde{s}_t)\bar{i},$$

where $\tilde{s}_t = \tilde{\epsilon}$ if $s_t = 0$ and $\tilde{\epsilon}$ is some arbitrarily small positive constant; and $\tilde{s}_t = 1$ otherwise. We introduce $\tilde{s}_t$ to ensure that the inflation process is well defined, yielding the following Markov-switching expectational difference equation for inflation:

$$\pi_t = (\phi_\pi \tilde{s}_t)^{-1} E_t \pi_{t+1} + (\phi_\pi \tilde{s}_t)^{-1} \left((1 - \tilde{s}_t)\,\bar{i} + \sigma u_t\right), \tag{B1}$$

where $\phi_\pi > 0$ and $y_t = 0$ for all $t$ is imposed in the Taylor rule with flexible prices. From Cho (2021),[1] (B1) is determinate if and only if

$$r(F) = r \begin{pmatrix} p_{11}(\phi_\pi)^{-2} & p_{10}(\phi_\pi)^{-2} \\ p_{01}(\epsilon)^{-2} & p_{00}(\epsilon)^{-2} \end{pmatrix} < 1,$$

where $p_{10} = 1 - p_{11}$, $p_{01} = 1 - p_{00}$, $\epsilon = \phi_\pi \tilde{\epsilon}$, and $r(F)$ denotes the spectral radius of the matrix $F$. The eigenvalues of $F$, $\lambda_1$ and $\lambda_2$, are the roots of the following quadratic equation:

$$f(\lambda) = \lambda^2 - (p_{00}(\epsilon)^{-2} + p_{11}(\phi_\pi)^{-2})\lambda + (p_{11} + p_{00} - 1)\phi_\pi^{-2}\epsilon^{-2} = 0.$$

[1] See Appendix D for further details.

As demonstrated on p. 28 of LaSalle (1986), both eigenvalues, $\lambda_1$ and $\lambda_2$, are inside the unit circle if and only if

$$\begin{aligned} |(p_{11}+p_{00}-1)\phi_\pi^{-2}\epsilon^{-2}| &< 1, \\ |p_{00}\epsilon^{-2}+p_{11}\phi_\pi^{-2}| &< 1+(p_{11}+p_{00}-1)\phi_\pi^{-2}\epsilon^{-2}. \end{aligned}$$

The first condition for determinacy, $|(p_{11}+p_{00}-1)\phi_\pi^{-2}\epsilon^{-2}| < 1$, is surely violated for $\phi_\pi < \infty$ as $\epsilon \to 0$. Hence, the model (B1) is indeterminate when $\tilde{\epsilon} \approx 0$.

From McClung (2020), we obtain E-stability of the MSV solution to (B1) if

$$r^e(A) = r^e \begin{pmatrix} p_{11}(\phi_\pi)^{-1}-1 & p_{10}(\phi_\pi)^{-1} \\ p_{01}(\epsilon)^{-1} & p_{00}(\epsilon)^{-1}-1 \end{pmatrix} < 0,$$

where $r^e(A)$ denotes the maximum of the real parts of the eigenvalues of $A$. Because the trace of $A$, $tr(A) = p_{00}(\epsilon)^{-1} + p_{11}(\phi_\pi)^{-1} - 2 > 0$ for small $\tilde{\epsilon}$, at least one eigenvalue of $A$ is positive as $\epsilon$ approaches zero. Hence, the MSV solution is E-unstable.

## *B.2. Proof of Proposition 2*

From Cho (2021), determinacy obtains if and only if

$$r(F) = r \begin{pmatrix} p_{11}(1+\phi_p)^{-2} & p_{10}(1+\phi_p)^{-2} \\ p_{01} & p_{00} \end{pmatrix} < 1,$$

where $p_{10} = 1 - p_{11}$, $p_{01} = 1 - p_{00}$, and $r(F)$ denotes the spectral radius of the matrix $F$. The eigenvalues of $F$, $\lambda_1$ and $\lambda_2$, are the roots of the following quadratic equation:

$$f(\lambda) = \lambda^2 - (p_{00} + p_{11}(1+\phi_p)^{-2})\lambda + (p_{11}+p_{00}-1)(1+\phi_p)^{-2} = 0.$$

As demonstrated on p. 28 of LaSalle (1986), both eigenvalues, $\lambda_1$ and $\lambda_2$, are inside the unit circle if and only if

$$\begin{aligned} |(p_{11}+p_{00}-1)(1+\phi_p)^{-2}| &< 1, \\ |p_{00}+p_{11}(1+\phi_p)^{-2}| &< 1+(p_{11}+p_{00}-1)(1+\phi_p)^{-2}, \end{aligned}$$

which holds provided that $p_{00}+p_{11}-1>0$, $\phi_p>0$, and $p_{00}<1$. From McClung (2020), E-stability of the MSV solution is obtained if

$$r^e(A) = r^e\begin{pmatrix} p_{11}(1+\phi_p)^{-1}-1 & p_{10}(1+\phi_p)^{-1} \\ p_{01} & p_{00}-1 \end{pmatrix} < 0,$$

where $r^e(A)$ denotes the maximum of the real parts of the eigenvalues of $A$. Because the trace of $A$ is negative (i.e., $tr(A) = p_{11}(1+\phi_p)^{-1}+p_{00}-2<0$), and the determinant of $A$ is positive (i.e., $det(A) = (1-p_{00})(1-1/(1+\phi_p))>0$) under the assumptions in Proposition 2, both eigenvalues of $A$ have negative real parts.

## *B.3. Proof of Proposition 3*

First, obtain the model solution for $t \geq T$. For $t \geq T$, the model assumes the form:

$$x_t = A^* E_t x_{t+1} + B^* x_{t-1},$$

where $(A^*, B^*) = (A(1), B(1))$ and $(A(1), B(1))$ are defined in the main text (i.e., see (A1)).[2] If $\phi_\pi$ and $\phi_y$ are sufficiently large (e.g., $\phi_\pi > 1$ under IT or $\phi_p > 0$ under PLT), then the

[2]Note that $C^* = C(1) = 0$ in this case. Also recall that we set $v_t = 0$ for simplicity, which allows us to write $D^* = D(0) = D(1) = 0$, but results are not sensitive to this assumption.

unique REE for $t \geq T$ assumes the form

$$\begin{aligned} x_t &= \Omega^* x_{t-1}, \\ \Omega^* &= (I - A^*\Omega^*)^{-1} B^*. \end{aligned} \tag{B2}$$

One can easily show that (B2) is the unique REE law of motion for $x_t$, $t \geq T$ using a variety of standard linear rational expectation techniques.[3] For $t = 0, \ldots, T-1$ the model is given by

$$x_t = \begin{cases} A^* E_t x_{t+1} + B^* x_{t-1} & \text{if } \; s_t = 1 \\ A(0) E_t x_{t+1} + B(0) x_{t-1} + C(0) & \text{if } \; s_t = 0 \end{cases} \tag{B3}$$

where $A(0), B(0), C(0)$ are also defined in the main text. Let $x_t^j = x_t | s_t = j$ for $j = 0, 1$. It is immediately apparent that $x_t^1 = \Omega^* x_{t-1}$ for all $t$. For $t = T-1$ we have

$$\begin{aligned} x_{T-1}^0 &= A(0) E_{T-1}(x_T | s_{T-1} = 0) + B(0) x_{T-2}^0 + C(0) \\ &= A(0)\Omega^* x_{T-1}^0 + B(0) x_{T-2}^0 + C(0) \\ &= (I - A(0)\Omega^*)^{-1} B(0) x_{T-2}^0 + (I - A(0)\Omega^*)^{-1} C(0) \\ &= \Omega(0)_{T-1} x_{T-2}^0 + \Gamma(0)_{T-1}. \end{aligned}$$

Therefore we have

$$\begin{aligned} x_{T-2}^0 &= (I - A(0)(p_{00}\Omega(0)_{T-1} + (1-p_{00})\Omega^*))^{-1} B(0) x_{T-3}^0 \\ &+ (I - A(0)(p_{00}\Omega(0)_{T-1} + (1-p_{00})\Omega^*))^{-1} (A(0) p_{00} \Gamma(0)_{T-1} + C(0)) \\ &= \Omega(0)_{T-2} x_{T-3}^0 + \Gamma(0)_{T-2}. \end{aligned}$$

[3] Standard linear RE techniques apply to this model class because of its linear form for $t \geq T$. See Cagliarini and Kulish (2013), Kulish and Pagan (2017) and Gibbs and McClung (forthcoming) for more.

Proceeding recursively backward in time:

$$\begin{aligned}
\Omega(0)_t &= (I - A(0)(p_{00}\Omega(0)_{t+1} + (1-p_{00})\Omega^*))^{-1}B(0), \\
\Gamma(0)_t &= (I - A(0)(p_{00}\Omega(0)_{t+1} + (1-p_{00})\Omega^*))^{-1}\left(A(0)p_{00}\Gamma(0)_{t+1} + C(0)\right),
\end{aligned}$$

from $t = T-2$ to $t = 0$. The solution for $t \geq 0$ is therefore given by:

$$x_t = \begin{cases} \Omega(0)_t x_{t-1} + \Gamma(0)_t & \text{if } \ s_t = 0 \\ \Omega^* x_{t-1} & \text{if } \ s_t = 1 \ \text{ or } \ t \geq T. \end{cases} \tag{B4}$$

Define $F(0)_t$ as follows:

$$F(0)_t = (I - A(0)(p_{00}\Omega(0)_{t+1} + (1-p_{00})\Omega^*))^{-1}A(0)$$

for $t = 0, \ldots, T-2$, and

$$F(0)_{T-1} = (I - A(0)\Omega^*)^{-1}A(0).$$

Next, we recast the model and model solution in the form (A1) and (A2), respectively, and assess mean-square stability, E-stability, and uniqueness of the REE (B4). Let $\tilde{s}_t \in \{0, 1, \ldots, T\}$ denote a $T+1$-state Markov process with transition matrix:

$$\tilde{P} = \begin{pmatrix} 0 & p_{00} & 0 & \ldots & 0 & 1-p_{00} \\ 0 & 0 & p_{00} & \ldots & 0 & 1-p_{00} \\ \vdots & & & \ddots & & \vdots \\ 0 & \ldots & & & p_{00} & 1-p_{00} \\ 0 & \ldots & & & 0 & 1 \\ 0 & \ldots & & & 0 & 1 \end{pmatrix}.$$

Let $\tilde{s}_t = t$ if $0 \leq t < T$ *and* $s_t = 0$; otherwise, $\tilde{s}_t = T$.[4] This implies restrictions on the $(i,j)$-element of $\tilde{P}$, $\tilde{P}_{ij}$: $\tilde{P}_{ij} = Pr(\tilde{s}_{t+1} = j|\tilde{s}_t = i) = p_{00}$ if $j \leq T-1$ and $i = j-1 \geq 0$; $\tilde{P}_{iT} = 1 - p_{00}$ for $0 \leq i < T-1$ and $\tilde{P}_{iT} = 1$ for $i \geq T-1$; otherwise, $\tilde{P}_{ij} = 0$. Similarly, let $(A(\tilde{s}_t), B(\tilde{s}_t), C(\tilde{s}_t)) = (A(0), B(0), C(0))$, $\tilde{\Omega}(\tilde{s}_t) = \Omega(0)_{\tilde{s}_t}$, $\tilde{\Gamma}(\tilde{s}_t) = \Gamma(0)_{\tilde{s}_t}$ and $\tilde{F}(\tilde{s}_t) = F(0)_{\tilde{s}_t}$ for $\tilde{s}_t = 0, \ldots, T-1$ and $(A(T), B(T), C(T)) = (A^*, B^*, 0)$, $\tilde{\Omega}(T) = \Omega^*$, $\tilde{\Gamma}(T) = 0$ and $\tilde{F}(T) = F^* = (I - A^*\Omega^*)^{-1}A^*$. The model and solution are now in the form (A1) and (A2), respectively. Following Costa, Fragoso and Marques (2005) and Cho (2021), one can show that the solution (B4) is mean-square stable if and only if $r(\Omega^* \otimes \Omega^*) < 1$, where $r(A)$ denotes the spectral radius of $A$. Equivalently:

$$
\begin{aligned}
r(\bar{\Psi}_{\tilde{\Omega}\otimes\tilde{\Omega}}) &= r\begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 \\
p_{00}\tilde{\Omega}(1)\otimes\tilde{\Omega}(1) & 0 & 0 & \ldots & 0 & 0 \\
0 & p_{00}\tilde{\Omega}(2)\otimes\tilde{\Omega}(2) & 0 & \ldots & 0 & 0 \\
\vdots & & & \ddots & & \vdots \\
0 & \ldots & & & 0 & 0 \\
(1-p_{00})\Omega^*\otimes\Omega^* & (1-p_{00})\Omega^*\otimes\Omega^* & \ldots & & \Omega^*\otimes\Omega^* & \Omega^*\otimes\Omega^*
\end{pmatrix} \\
&= r\left(\Omega^*\otimes\Omega^*\right) < 1.
\end{aligned}
$$

We have $r(\Omega^* \otimes \Omega^*) < 1$ by the fact that (B2) is the unique REE law of motion for $x_t$, $t \geq T$.

Next, consider uniqueness and let $\hat{x}_t$ denote an arbitrary solution of the model. Then $\hat{x}_t$ can be expressed as: $\hat{x}_t = \tilde{\Omega}(\tilde{s}_t)\hat{x}_{t-1} + \tilde{\Gamma}(\tilde{s}_t) + w_t$ where $w_t = \tilde{F}(\tilde{s}_t)E_t w_{t+1}$. Since (B2) is the unique mean-square stable solution for $\tilde{s}_t = T$ (i.e., $r(F^* \otimes F^*) < 1$), $w_t = 0$ if $\tilde{s}_t = T$. This implies $w_t = 0$ for $\tilde{s}_t \in \{0, \ldots, T-1\}$ which can be verified by substituting $E_t x_{t+1} = E_t \hat{x}_{t+1}$ into the model and solving for $\hat{x}_t$ and $w_t$. Therefore, (B4) is the unique mean-square stable

[4] Note that to recover the specific model under consideration we impose the restriction: $\tilde{s}_0 \in \{0, T\}$. However, the proof of Proposition 3 applies to the more general case: $\tilde{s}_0 \in \{0, 1, \ldots, T\}$. We furthermore note that there exists a unique stationary probability distribution, $\bar{\pi}\tilde{P} = \bar{\pi}$, such that $\lim_{t\to\infty} \pi_t = \bar{\pi}$ given any $\pi_0$ where the $i$th element of $\pi_0$ is the probability that $\tilde{s}_0 = i$.

solution of (B3). Alternatively, following Cho (2021), we have

$$\begin{aligned} r(\Psi_{\tilde{F}\otimes\tilde{F}}) &= r\begin{pmatrix} 0 & p_{00}\tilde{F}(0)\otimes\tilde{F}(0) & 0 & \dots & 0 & (1-p_{00})\tilde{F}(0)\otimes\tilde{F}(0) \\ 0 & 0 & p_{00}\tilde{F}(1)\otimes\tilde{F}(1) & \dots & 0 & (1-p_{00})\tilde{F}(1)\otimes\tilde{F}(1) \\ \vdots & & & \ddots & & \vdots \\ 0 & \dots & & & 0 & \tilde{F}(T-1)\otimes\tilde{F}(T-1) \\ 0 & \dots & & & 0 & F^*\otimes F^* \end{pmatrix} \\ &= r\left(F^*\otimes F^*\right) < 1, \end{aligned}$$

where $r(F^*\otimes F^*) < 1$ again follows from the fact that (B2) is the unique REE law of motion for $x_t$ for $t \geq T$ (e.g., see Cho (2021) or Appendix D). E-stability follows from $r(\bar{\Psi}_{\tilde{\Omega}\otimes\tilde{\Omega}}) < 1$ and $r(\Psi_{\tilde{F}\otimes\tilde{F}}) < 1$ by Propositions 1 and 2 of McClung (2020).

# APPENDIX C: RATIONAL EXPECTATIONS EQUILIBRIUM

As explained in section 2 of the main text, rational agents are assumed to possess complete, homogeneous information of the economy and form true mathematical expectations, $E_t x_{t+1}$, conditional on complete time-$t$ information.[5] A rational expectations solution is any stochastic process $\{x_t\}$ that solves the model (A1) under the above-mentioned assumptions. In general, there can be two types of solutions: (i) minimal state variable (MSV) solutions, which express $x_t$ as a function of fundamental predetermined variables, $x_{t-1}$, $C(s_t)$, $s_t$, and no other variables; and (ii) non-fundamental (sunspot) solutions, which express $x_t$ as a function of $x_{t-1}$, $C(s_t)$, $s_t$, and extraneous variables that do not appear in (A1). An REE is a mean-square stable rational expectations solution.[6] *If* a unique REE of (A1) exists, then it

[5] Note that agents do not know $s_{t+j}$ for any $j \geq 1$ at time $t$.

[6] See the main paper for details about the literature on mean-square stability and alternative stability and solution concepts.

assumes the MSV form (A2) where

$$\Omega(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Omega(s_{t+1})\right)^{-1} B(s_t), \tag{C1}$$

$$Q(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Omega(s_{t+1})\right)^{-1} D(s_t), \tag{C2}$$

$$\Gamma(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Omega(s_{t+1})\right)^{-1} \left(C(s_t) + A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Gamma(s_{t+1})\right). \tag{C3}$$

We apply the forward method of Cho (2016) to obtain a solution of the form (A2), and use the determinacy conditions in Cho (2016) and Cho (2021), which are tractable conditions that depend only on $A(s_t), B(s_t), \Omega(s_t)$, $p_{00}$, and $p_{11}$. These determinacy conditions, when satisfied, ensure that the MSV solution (A2) is the unique REE of the model (A1). Thus, if a given MSV solution (A2) satisfies the conditions in Cho (2021), then all non-fundamental solutions of (A1) and all other MSV solutions of (A1) are mean-square *unstable*. If the determinacy conditions fail, and the MSV solution we obtain is mean-square stable, then we have indeterminacy.[7] In order to apply the solution approach and determinacy conditions of Cho (2016) and Cho (2021) to a model of the form (A1), which contains a regime-switching intercept term, we need to make slight modifications to the model (see Appendix D below).

# APPENDIX D: REGIME-SWITCHING MODEL WITH INTERCEPT

The model (A1) contains a regime-switching intercept term, $C(s_t)$. Although recent works (e.g., Bianchi and Melosi (2017)) have solved Markov-switching DSGE models with regime-switching intercept terms, those that discuss the determinacy properties of Markov-switching models typically assume $C(s_t) = 0$, for all $s_t$. Here, we show one way to handle the intercept

[7] We always obtain at least one mean-square stable solution for any calibration of the model we consider in this study. Therefore, we can always determine whether our model is determinate or indeterminate.

term when solving the model using the forward method of Cho (2016), and we argue that the determinacy conditions of Cho (2021) can be applied to a model with $C(s_t) \neq 0$. Throughout the appendix we assume $v_t = 0$ but the same result obtains if $v_t \neq 0$.

### *Appendix D.1. Solution Method*

The model (A1) assumes the form:

$$x_t = A(s_t)E_t x_{t+1} + B(s_t)x_{t-1} + C(s_t).$$

where $s_t$ is an $S+1$-state first-order Markov process with transition matrix, $P$. The elements of $P$ are defined as $p_{ij} = Pr(s_{t+1} = j|s_t = i)$. As shown below, if a unique REE exists then it assumes the MSV form:

$$x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t).$$

We use the forward method of Cho (2016) to obtain a solution for $\Omega(s_t)$. The intercept term, $\Gamma(s_t)$, must satisfy (C3). Define $F(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Omega(s_{t+1})\right)^{-1} A(s_t)$, $G(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Omega(s_{t+1})\right)^{-1} C(s_t)$, $G = (G(0)', G(1)', \ldots, G(S)')'$, $\Gamma = (\Gamma(0)'$ $\Gamma(1)', \ldots, \Gamma(S)')'$, and

$$\Psi_F = \begin{pmatrix} p_{00}F(0) & \ldots & p_{0S}F(0) \\ \vdots & \ddots & \vdots \\ p_{S0}F(S) & \ldots & p_{SS}F(S) \end{pmatrix}.$$

Then, given $\Omega(s_t)$, the solution for $\Gamma$ is unique and given by

$$\Gamma = (I - \Psi_F)^{-1} G$$

assuming $(I-\Psi_F)$ is non-singular.[8]

### *Appendix D.2. Determinacy*

Consider the class of Markov-switching DSGE models given by (A1) assuming $C(s_t) = 0$ for all $s_t$:

$$x_t = A(s_t)E_t x_{t+1} + B(s_t)x_{t-1}. \tag{D1}$$

From Cho (2016), we can express any rational expectations solution of (D1) as

$$x_t = \Omega(s_t)x_{t-1} + w_t, \tag{D2}$$

$$w_t = F(s_t)E_t w_{t+1}, \tag{D3}$$

where $\Omega(s_t)$ satisfies (C1), and

$$F(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Omega(s_{t+1})\right)^{-1} A(s_t).$$

Thus, any solution of (D1) can be represented as the sum of an MSV component, $x_t = \Omega(s_t)x_{t-1}$, and a non-fundamental process, $w_t$. Following Cho (2021), an REE of (D1) is any mean-square stable rational expectations solution satisfying (D2) where $w_t$ is a mean-square stable solution of (D3) and satisfies the explicit restrictions on solutions of (D3) stated in Cho (2021). Also following Cho (2021), we say that (D1) is determinate if (a) (D2) with $w_t = 0$ is the law of motion for the unique mean-square stable MSV solution of (D1); (b) a non-zero mean-square stable solution of (D3) associated to the unique stable MSV solution which satisfies restrictions stated in Cho (2021) does not exist. To assess determinacy, consider the

[8] If $(I-\Psi_F)$ is singular, then $\Psi_F$ has a unit eigenvalue. McClung (2020) shows that the real parts of $\Psi_F$ must be less than one for the underlying equilibrium to be E-stable (in our numerical analysis, the eigenvalues of $\Psi_F$ are always inside the unit circle when E-stability is satisfied). Furthermore, we can show that the underlying model is indeterminate if $r(\Psi_F) > 1$. Thus, we do not encounter singular $(I-\Psi_F)$ in cases where the REE is E-stable or the model is determinate.

following matrices:

$$\bar{\Psi}_{\Omega\otimes\Omega} = \begin{pmatrix} p_{00}\Omega(0)\otimes\Omega(0) & \dots & p_{S0}\Omega(0)\otimes\Omega(0) \\ \vdots & \ddots & \vdots \\ p_{0S}\Omega(S)\otimes\Omega(S) & \dots & p_{SS}\Omega(S)\otimes\Omega(S) \end{pmatrix}$$
$$\Psi_{F\otimes F} = \begin{pmatrix} p_{00}F(0)\otimes F(0) & \dots & p_{0S}F(0)\otimes F(0) \\ \vdots & \ddots & \vdots \\ p_{S0}F(S)\otimes F(S) & \dots & p_{SS}F(S)\otimes F(S) \end{pmatrix}.$$

Theorem 1 gives the determinacy criterion for (D1), under the assumptions made explicit in Cho (2021), which we maintain throughout this paper.

**Theorem 1** *Consider the model (D1) and suppose $\Omega(s_t)$ exists and is real-valued. Then, (D1) is a determinate model if and only if:*

1. $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$
2. $r(\Psi_{F\otimes F}) \leq 1$.

**Proof:** see Propositions 1 and 2 in Cho (2021). ■.

Intuitively, $r(\Psi_{F\otimes F}) \leq 1$ ensures that $w_t = 0$ in an REE and that $\Omega(s_t)$ is the only fixed point of (C1) that gives a mean-square stable solution of (D1); and $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ ensures that the MSV solution, $x_t = \Omega(s_t)x_{t-1}$, is mean-square stable. Hence, if $r(\Psi_{F\otimes F}) \leq 1$ and $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ then the MSV solution, $x_t = \Omega(s_t)x_{t-1}$, is the unique mean-square stable solution of (D1).[9] If $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ and $r(\Psi_{F\otimes F}) > 1$ then there are multiple REE.

We examine a closely-related model, given by (A1), which is reproduced here:

$$x_t = A(s_t)E_t x_{t+1} + B(s_t)x_{t-1} + C(s_t).$$

[9] Note that Theorem 1 technically applies to a particular MSV solution ("minimum-of-modulus" (MOD) solution). However, if $r(\Psi_{F\otimes F}) \leq 1$ and $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ then the MSV solution, $x_t = \Omega(s_t)x_{t-1}$, is the MOD solution. Moreover, if $B(s_t) = 0$, then the MSV is unique (and therefore the MOD solution by default). See Cho (2021) for more information.

Following Farmer, Waggoner and Zha (2011), we can recast the model in the form

$$\bar{x}_t = \bar{A}(s_t)E_t\bar{x}_{t+1} + \bar{B}(s_t)\bar{x}_{t-1}, \tag{D4}$$

where $\bar{x}_t = (x_t', z_t)'$, $z_t$ is a dummy variable satisfying $z_0 = 1$ and $z_t = z_{t-1}$ for all $t$, and

$$\bar{A}(s_t) = \begin{pmatrix} A(s_t) & 0_{n\times 1} \\ 0_{1\times n} & 0 \end{pmatrix}$$
$$\bar{B}(s_t) = \begin{pmatrix} B(s_t) & C(s_t) \\ 0_{1\times n} & 1 \end{pmatrix}.$$

Given the form of (D4) and following Cho (2016), any REE of (D4) (and therefore any REE of (A1)) can be expressed as

$$\begin{aligned} \bar{x}_t &= \bar{\Omega}(s_t)\bar{x}_{t-1} + \bar{w}_t, && \text{(D5)} \\ \bar{w}_t &= \bar{F}(s_t)E_t\bar{w}_{t+1}, && \text{(D6)} \end{aligned}$$

where

$$\begin{aligned} \bar{\Omega}(s_t) &= \left(I - \bar{A}(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\bar{\Omega}(s_{t+1})\right)^{-1}\bar{B}(s_t), \\ \bar{F}(s_t) &= \left(I - \bar{A}(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\bar{\Omega}(s_{t+1})\right)^{-1}\bar{A}(s_t). \end{aligned}$$

Given $z_t = z_{t-1}$ for all $t$ and $z_0 = 1$, and the restrictions on $\bar{B}(s_t)$ and $\bar{A}(s_t)$, one can now recast the solution (D5) and (D6), and therefore any rational expectations solution of (A1), as:

$$x_t = \Gamma(s_t) + \Omega(s_t)x_{t-1} + w_t, \tag{D7}$$

where $w_t$ satisfies (D3) and $\Omega(s_t)$ and $\Gamma(s_t)$ satisfy (C1) and (C3), respectively.

Following Cho (2021), an REE of (A1) is any mean-square stable rational expectations solution satisfying (D7) where $w_t$ is a mean-square stable solution of (D3) and satisfies the explicit restrictions on solutions of (D3) stated in Cho (2021). Also following Cho (2021), we say that (A1) is determinate if (a) (D7) with $w_t = 0$ is the law of motion for the unique mean-square stable MSV solution of (A1); (b) a non-zero mean-square stable solution of (D3) associated to the unique stable MSV solution which satisfies restrictions stated in Cho (2021) does not exist. Consider $r(\Psi_{F\otimes F})$ and $r(\bar{\Psi}_{\Omega\otimes\Omega})$ from above. Then by applying results from Costa, Fragoso and Marques (2005) and Cho (2021), one can show that the MSV process (D7) is the unique mean-square stable if and only if $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$, and $r(\Psi_{F\otimes F}) \leq 1$. If $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ and $r(\Psi_{F\otimes F}) > 1$ then there are multiple REE.[10]

# APPENDIX E: ADAPTIVE LEARNING FRAMEWORK AND E-STABILITY

This appendix provides additional details about the adaptive learning framework which are helpful for understanding the E-stability condition. To derive the E-stability conditions we first establish the information set, $\mathcal{I}_t$, available to adaptive learning agents when forming expectations at time $t$. The baseline assumption in our paper is that agents have "contemporaneous information": $(P, x_t, s_t, v_t) \in \mathcal{I}_t$. We also assume that learning agents recursively estimate the coefficients, $(a(s_t), b(s_t), c(s_t))$, of the PLM (A3), as explained in section 2.3 of the manuscript. It is important to recognize a couple of facts about the agents' PLM and our model of adaptive learning. First, (A3) is correctly specified (i.e. the PLM has the functional form of the MSV solution) but the agents do not know the MSV coefficients and must learn about them recursively. Second, because agents know $s_t$, they do not need so-

[10] In our applications, cases where $\Psi_{F\otimes F}$ of $\Psi_F$ (as defined above) have a unit eigenvalue are not encountered or addressed.

phisticated Markov-switching VAR techniques to estimate the coefficients of (A3); they need only estimate two linear models in real time (one for each regime) using standard techniques such as the least squares method.

Under the assumption of contemporaneous information, agents form expectations in real time as follows:[11]

$$\hat{E}_t x_{t+1} = \sum_{s_{t+1}} p_{s_t s_{t+1}} \left\{ a(s_{t+1})_{t-1} + b(s_{t+1})_{t-1} x_t \right\},$$

where $a(s_t)_{t-1}$, $b(s_t)_{t-1}$ and $c(s_t)_{t-1}$ denote agents' estimates of $a(s_t)$, $b(s_t)$ and $c(s_t)$, respectively, using all information available at the end of $t-1$. In what follows, we suppress the $t-1$ subscripts in the agents' PLM and let $(a(s_t), b(s_t), c(s_t))$ represent $(a(s_t)_{t-1}, b(s_t)_{t-1}, c(s_t)_{t-1})$. If agents make decisions contingent on these forecasts (i.e., if we substitute $\hat{E}_t x_{t+1}$ into (A1)), then the equilibrium at time $t$ is given by

$$\begin{aligned} x_t &= \left( I - A(s_t) \sum_{s_{t+1}} p_{s_t s_{t+1}} b(s_{t+1}) \right)^{-1} \left( B(s_t) x_{t-1} + D(s_t) v_t \right) \\ &+ \left( I - A(s_t) \sum_{s_{t+1}} p_{s_t s_{t+1}} b(s_{t+1}) \right)^{-1} \left( C(s_t) + A(s_t) \sum_{s_{t+1}} p_{s_t s_{t+1}} a(s_{t+1}) \right). \end{aligned} \tag{E1}$$

After observing $x_t$, agents update their estimates of $a(s_t)$, $b(s_t)$ and $c(s_t)$ using standard techniques such as least squares, holding fixed their beliefs about $a(j)$, $b(j)$, $c(j)$ where $j \neq s_t$. Note that Appendix F and McClung (2020) provide algorithms for implementing the recursive estimation of $a(s_t)$, $b(s_t)$ and $c(s_t)$. From (E1), it is apparent that the agents'

[11] Here, and throughout the paper, we use $\hat{E}_t$ to denote (potentially) non-rational expectations formed under adaptive learning. $E_t$ denote rational expectations.

beliefs, $a(s_t)$, $b(s_t)$, and $c(s_t)$, are self-confirming only if

$$
\begin{aligned}
b(s_t) &= \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}} b(s_{t+1})\right)^{-1} B(s_t), \\
c(s_t) &= \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}} b(s_{t+1})\right)^{-1} D(s_t), \\
a(s_t) &= \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}} b(s_{t+1})\right)^{-1} \left(C(s_t) + A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}} a(s_{t+1})\right).
\end{aligned}
$$

Note that these conditions are identical to (C1)-(C3). Therefore, learning agents' beliefs only converge to self-confirming values under adaptive learning with correctly-specified PLM (A3) if the agents learn the coefficients of an REE (A2). Formally, we say that adaptive learning agents learn the MSV solution (A2) if $(a(s_t), b(s_t), c(s_t)) \to (\Gamma(s_t), \Omega(s_t), Q(s_t))$ as $t \to \infty$. Following Proposition 1 of McClung (2020), we present the E-stability conditions that predict if agents can learn a solution (A2) by estimating (A3) in real time and making forecasts based on these estimates and contemporaneous information. When the E-stability conditions fail (i.e., "E-instability" obtains), agents will generally not learn the MSV solution.

For the purpose of presenting the E-stability conditions, consider (A2) and define

$$
F(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}} \Omega(s_{t+1})\right)^{-1} A(s_t),
$$

where all matrices are from the model (A1) and the model equilibrium (A2) under study. Under the contemporaneous information assumptions discussed in section 2, (A2) is E-stable if the real parts of the eigenvalues of

$$
\begin{aligned}
\Psi_{\Omega' \otimes F} &= \begin{pmatrix} p_{00}\Omega(0)' \otimes F(0) & p_{01}\Omega(0)' \otimes F(0) \\ p_{10}\Omega(1)' \otimes F(1) & p_{11}\Omega(1)' \otimes F(1) \end{pmatrix} \\
\Psi_F &= \begin{pmatrix} p_{00}F(0) & p_{01}F(0) \\ p_{10}F(1) & p_{11}F(1) \end{pmatrix}
\end{aligned}
$$

are less than one. See Proposition 1 of McClung (2020) for a formal proof.[12]

# APPENDIX F: E-STABILITY AND CONVERGENCE TO REE

This appendix demonstrates numerically that E-stability can predict convergence of the learning equilibrium law of motion to the mean-square stable REE law of motion. As discussed in the main text, deflationary spirals do not occur if the learning equilibrium law of motion converges to (A2) (i.e., because $lim_{t\to\infty}E_0\pi_t$ is finite in a mean-square stable equilibrium).

To demonstrate convergence in practice, we suppose agents update $\Phi(k)_t = (a(k)_t, b(k)_t, c(k)_t)'$ using the recursive estimator

$$\Phi(k)_t = \Phi(k)_{t-1} + \psi(k)_t R(k)_t^{-1} z_t (x_t - \Phi(k)'_{t-1} z_t)', \tag{F1}$$

$$R(k)_t = R(k)_{t-1} + \psi(k)_t (z_t z_t' - R(k)_{t-1}), \tag{F2}$$

where $z_t = (1\ x_t'\ v_t')'$, $\psi(k)_t = 1/t_k^\alpha$ if $s_t = k$ and 0 otherwise, $t_k$ is the number of periods such that $s_t = k$, $\alpha \in (0,1]$ and $k = 0, 1$.[13] Intuitively, (F1)-(F2) is a recursive (weighted) least squares estimator of the two linear regime-dependent PLMs.

Under contemporaneous information,[14] time-$t$ equilibrium is determined as follows.

Step 1 At the end of $t-1$, agents update $\Phi(k)_{t-1}$ using time-$t-1$ information and (F1)-(F2).

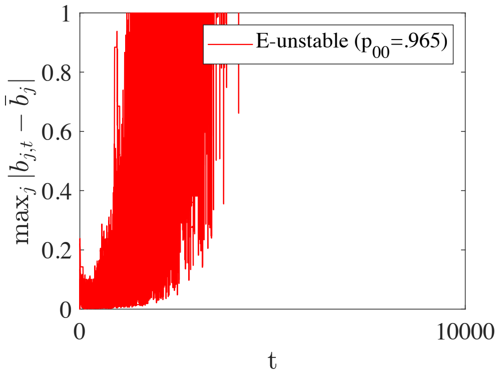
Step 2 At time-$t$, temporary equilibrium is given by substituting $\Phi(k)_{t-1}$ into (E1).

[12] Note that the proof of Proposition 1 of McClung (2020) assumes $C(s_t) = 0$. However, it is straightforward to show that $C(s_t) \neq 0$ does not affect any E-stability computations in the proof. Therefore, the value of the regime-switching intercept is irrelevant to E-stability.

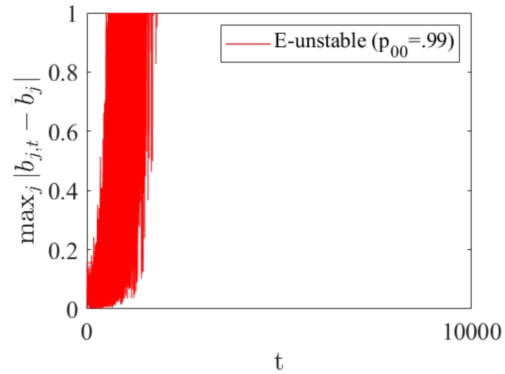
[13] We also consider cases with constant gain (i.e. $\psi(k)_t = \psi \in (0,1]$) as noted below.

[14] The qualitative results reported in this section hold if agents do not contemporaneously observe $x_t$.

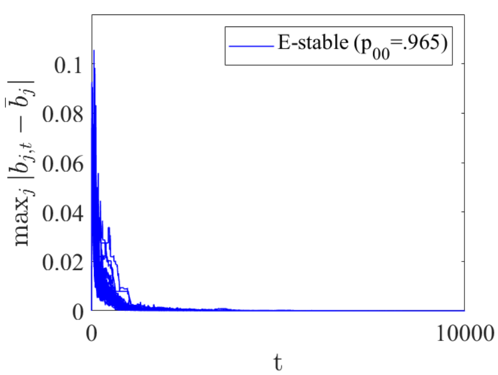
Figure A1: Learning and Convergence to the REE



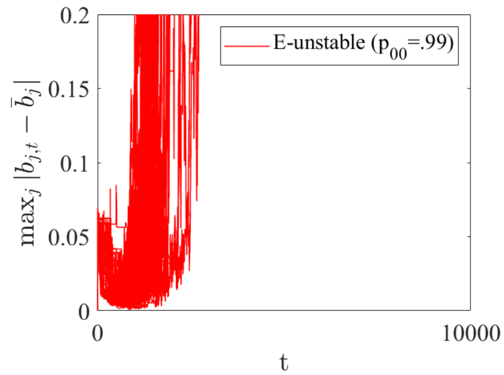

(a) Inflation Targeting ($p_{00} = 0.965$)



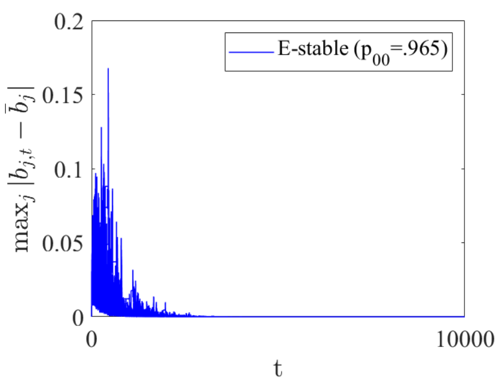

(b) Inflation Targeting ($p_{00} = 0.99$)



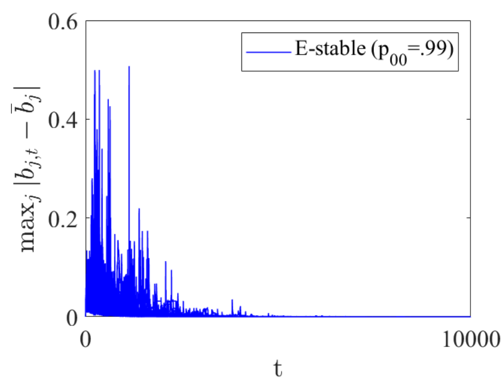

(c) Avg. Inflation Targeting ($p_{00} = 0.965$)




(d) Avg. Inflation Targeting ($p_{00} = 0.99$)




(e) Price Level Targeting ($p_{00} = 0.965$)




(f) Price Level Targeting ($p_{00} = 0.99$)

Note: The convergence results for various targeting rules are plotted for simulations. Convergence occurs if $\max_j |b_{j,t} - \bar{b}_j| \to 0$ as $t$ increases where $b_{j,t}$ in (F1) is the agents' current estimate of the coefficient $\bar{b}_j$ in the REE. The transition probability of $p_{00}$ indicates the probability that the economy remains at the ZLB in the next period. Average inflation targeting has the target window of $m = 9$ for all the simulations. Some plotted simulations feature flat line segments that are a consequence of off-equilibrium beliefs being fixed.

We can repeat Steps 1 and 2 to solve for temporary equilibrium at $t+1$ and so on. Figure A1 illustrates convergence to the REE law of motion in cases where E-stability conditions are satisfied, and divergence in cases where the conditions are not satisfied. Each panel of the figure illustrates the maximum distance between agents' current estimate of any of the $K$ coefficients of $\Phi(k)_t$ for $k = 0, 1$ (i.e., $b_{j,t}$ for $j = 1, \ldots, K$) and the true REE value of that coefficient (i.e., $\bar{b}_j$). Convergence occurs if $\max_j |b_{j,t} - \bar{b}_j| \to 0$ as $t$ increases. In each panel we assume that initial beliefs, $\Phi(k)_0$, are different from the true REE beliefs, and then simulate each model 50 times. Further, we set $p_{11} = 0.975$ and consider two different values of $p_{00}$ to generate both E-stable and E-unstable models. Particular attention is paid to $p_{00} = 0.965$, which ensures an expected ZLB duration equal to the duration of the 2008-2015 U.S. ZLB event.[15] The other value is given by $p_{00} = 0.99$, which leads to a longer expected duration of the ZLB than $p_{00} = 0.965$. All other parameters are set at the benchmark values as discussed in Section 4 of the manuscript. Across the simulations, we observe convergence to REE when E-stability is obtained and $\Phi(k)_0$ is sufficiently close to the REE, and divergence when E-stability conditions are not satisfied.

As depicted in Figure A1, the coefficients in the simulation with inflation targeting for both values of $p_{00}$ diverge away from their REE coefficients, whereas those for price level targeting converge to their REE values. This shows that price level targeting can promote real-time learning of REE more effectively than inflation targeting. In addition, the simulation for average inflation illustrates the importance of beliefs about transition probabilities; under $p_{00} = 0.965$ and average inflation targeting, beliefs converge to the REE values, but they diverge away from the REE if $p_{00} = 0.99$. The optimistic expectations of a shorter ZLB duration help promote E-stability. Following the same intuition, we find that the optimistic expectations (i.e., a shorter expected duration of the ZLB event such as $p_{00} = 0.965$) are

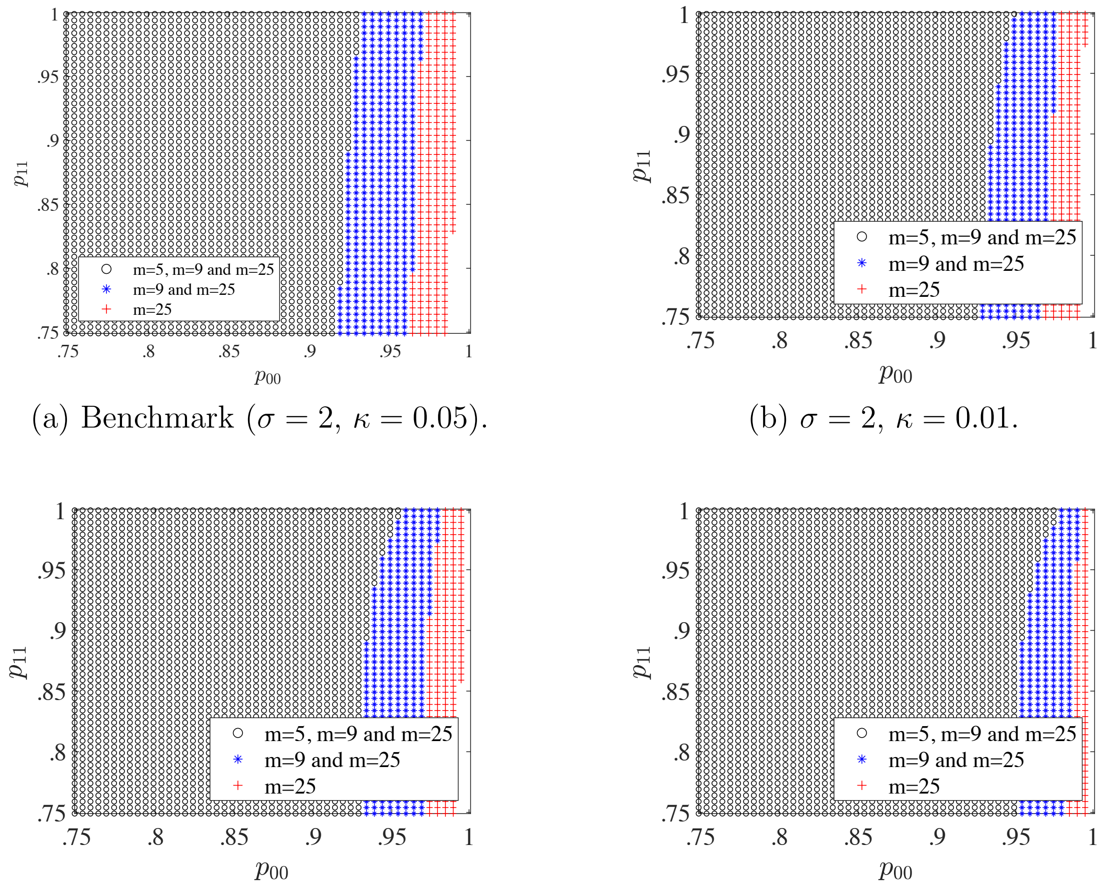
[15] The discrete-valued shocks are calibrated so that $E(i_t^*|s_t = 0) < -\bar{i}$ and $E(i_t^*|s_t = 1) > -\bar{i}$ where $i^*$ denotes the shadow rate in the REE. To keep the speed of learning high in the simulations, we impose $\psi(k)_t = \max\{1/t_k^{2/3}, 0.04\}$, though convergence would occur more gradually under the alternative assumption of decreasing gain ($\psi(k)_t = 1/t_k^{\alpha}$). Finally, we impose $v_{i,t} \sim \mathcal{N}(0, \sigma_i^2)$ where $i = s, d$ and $\sigma_d = \sigma_s = 0.001$ to ensure that $R(k)_t$ is nonsingular in simulations.

associated with a quicker convergence for price level targeting, and a slower divergence for inflation targeting compared to the cases involving a longer expected ZLB duration such as $p_{00} = 0.99$. These simulation results confirm our findings in Section 5.1.

# APPENDIX G: ROBUSTNESS OF AIT

Here we depict determinacy regions for alternative calibrations of the AIT rule. In particular, we recreate Figure 4 from the main text for different values of the structural parameters $m$ and $\phi_\pi$, assuming instead that $\phi_{\bar{\pi}} = \phi_\pi m$ (as opposed to $\phi_{\bar{\pi}} = \phi_\pi$ in Figure 4 from the main text). The purpose of the supplementary figure is to illustrate how the AIT results are sensitive to the averaging window ($m$), the policy rule coefficients, and deep structural parameters including $\sigma$ and $\kappa$.

Figure A2: Determinacy and AIT under Alternative Parameterizations




(a) Benchmark ($\sigma = 2$, $\kappa = 0.05$).

(b) $\sigma = 2$, $\kappa = 0.01$.

(c) $\sigma = 5$, $\kappa = 0.01$.

(c) $\sigma = 5$, $\kappa = 0.01$, $\phi_{\bar{\pi}} = 5$.

Note: For various parameterizations, the REE for an AIT rule is depicted with respect to $p_{00}$ and $p_{11}$ as follows. The white area depicts the indeterminacy region. This figure assumes the unweighted AIT rule with $\phi_{\bar{\pi}} = \phi_{\pi} m$. The model parameters are set at the benchmark values throughout this exercise unless noted otherwise.

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