ON ROBUSTNESS OF AVERAGE INFLATION TARGETING*

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Abstract

This paper considers average inflation targeting (AIT) policy in a New Keynesian model with adaptive learning agents. There are stability concerns regarding AIT when agents have imperfect knowledge and the averaging window length is not public knowledge. These stability risks near the inflation target steady state would likely be avoided under inflation targeting (IT) or price level targeting (PLT). Near the zero interest rate steady state, AIT under-performs PLT and does not necessarily outperform IT. Communicating the averaging window length or adopting an asymmetric average inflation target that judges below-target average inflation more negatively avoids these pitfalls.

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1 Introduction

In much of the 2008 -2021 period, central banks had to keep the policy interest rates at approximately zero level, popularly called the zero lower bound (ZLB) or the liquidity trap. The usual framework of inflation targeting became largely ineffective in the ZLB regime and central banks had to revert to unconventional monetary policies. The ZLB regime after the financial crisis inspired central bankers to think about alternatives to inflation targeting. Price level targeting (PLT) and more complex strategies that can deliver above-target "make-up" inflation after a period of low inflation at the ZLB were discussed. In

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2019 the Federal Reserve initiated a review of its monetary policy strategy. The review process culminated in August 2020 in the announcement by the Chairperson Powell (2020) that the policy framework of the Fed is to be based on Average Inflation Targeting (AIT).[1]

The research literature on AIT is still relatively small.[2] Most research papers studying AIT employ the rational expectations (RE) assumption, so that agents understand the basic structure of AIT policy. This assumption is hard to reconcile with recent studies on how expectations respond to AIT. Coibion, Gorodnichenko, Knotek II and Schoenle (2023) found that the Fed's August 2020 announcement of AIT had little effect on household inflation expectations. This result could suggest that households do not understand the basic structure and implications of AIT. Salle (2021) examined data from laboratory experiments involving AIT and finds that agents struggle to understand the lag structure implied by AIT. On the other hand, Hoffman, Moench, Pavlova and Schultefrankenfeld (2022) find that German households might understand the implications of an asymmetric AIT strategy.

This paper questions whether AIT can stabilize average inflation and anchor long-run expectations when key details of the policy regime are not public knowledge. To this end, we consider the performance of AIT in a standard New Keynesian (NK) model when private agents have imperfect knowledge of the economy and engage in learning to forecast its dynamics. It is assumed that when forming expectations private agents statistically estimate the laws of motion for the endogenous variables that they need to forecast. In each period private agents make optimal decisions given the current forecasts and the economy evolves as a sequence of temporary equilibria that are defined by forecasts and private decisions in each period. As time progresses, new data leads to updating of the forecast functions and new temporary equilibria.

This approach, called adaptive learning, relies on a more realistic model of expectations formation and decision making than rational or boundedly rational decision rules under RE.[3] As desideratum, the monetary policy framework should be able to guide both actual and expected inflation back to the target level following a shock to the economy. Convergence of the learning dynamics under different policy rules has been widely studied in the literature following the pioneering work in the early 2000's.[4]

In benign circumstances the economy reaches a long-run REE (RE equilibrium), but this depends on the structure of the economy and in particular the policy rule used by the central bank and the private sector's knowledge of the policy. As a starting point, in Section 2 we follow much of the nascent AIT literature and specify a transparent and symmetric L-period moving average inflation target. Under a transparent AIT regime, we find that the central bank can hit their long-run inflation target steady state using a simple AIT interest rate rule, provided that the economy is sufficiently close to the target when the AIT regime is introduced. However, in some cases the economy cannot reach a long-run REE if the averaging window is too long. Further, simulations from a fully nonlinear NK

[1]The European Central Bank also reformed its monetary policy framework and introduced a symmetric 2% inflation target in 2021.

[2]Nessen and Vestin (2005), Mertens and Williams (2019), Budianto, Nakata and Schmidt (2023) and Amano, Gnocchi, Leduc and Wagner (2020) is a representative sample of the papers studying basics of AIT. Andrade, Gali, Le Bihan and Matheron (2021) focuses on make-up strategies for monetary policy. Jia and Wu (2023) considers ambiguities in communication with AIT.

[3]For example, IMF World Economic Outlook, October 2022 uses adaptive learning as the baseline framework for modelling expectations formation. See IMF (2022), pp.63-66 and the Annex to the report.

[4]See e.g. Bullard and Mitra (2002) and Evans and Honkapohja (2003). Surveys of the subsequent literature are provided, for example, in Gaspar, Smets and Vestin (2010) and Woodford (2013).

model in Section 5 show that a transparent AIT policy does not necessarily outperform a standard IT policy in a liquidity trap. Transparency is also inconsistent with the Federal Reserve's opaque strategy, which is not based on a published measure of average inflation. See, e.g., the interview of John Williams in FT Live, November 13, 2020.

Our main framework (Section 3) additionally assumes that agents do not know the central banks's averaging window ("opacity") when learning to forecast. The AIT policy rule is assumed to satisfy the Taylor principle at the target equilibrium, but this is not part of agents' information set. In these circumstances the central bank that does not announce the average inflation target may likely fail to steer inflation to the inflation target following a shock to the economy. The instability of AIT is sensitive to the information in the specification of agents' forecasting models. Instability happens if the data window private agents use is too short, i.e. the number of lags in agents' forecasting model is smaller than the number of lags in the AIT policy rule. Moreover, the instability problem is exacerbated if prices are very flexible. In contrast, if agents' perceived data window is as long as or longer than the data window of the AIT policy rule, then after an exogenous shock the private agents can learn the number of lags used in the AIT rule (see Section 4), and the learning process can steer the economy back to the targeted RE equilibrium.

However, this requires that private agents are sophisticated econometricians. In practice agents are not likely to use a sufficient number of observations of lagged data when forming expectations. For empirics of inflation expectations, see e.g. Coibion, Gorodnichenko and Kamdar (2018) and Coibion, Gorodnichenko, Kumar and Pedemonte (2020). Furthermore, agents with a perceived window that matches or exceeds the true averaging window can fail to learn the REE when the true window length is sufficiently long. Thus, in all cases, there are risks to economic stability when the central bank symmetrically targets a simple moving average of inflation with long averaging window.

The central bank may avoid these pitfalls of a symmetric target by adopting an asymmetric policy that dramatically shortens the averaging window when inflation is running high. See Section 6. We illustrate the benefits of asymmetry by assuming that policy responds to the output gap and a moving average of inflation when average inflation is below target, but only responds to current inflation and output when average inflation is high. Other variations on models of AIT which shape stability risks to the economy, including weighted average schemes, are considered in Section 7, and directions for future work are discussed in Section 8.

These results call for a cautious approach to targeting an opaque measure of average inflation. Risks to the stability of inflation emerge if the averaging window is opaque and moderately lengthy. The central bank can minimize such risks by keeping the window fairly short and encouraging forecasters to adopt sophisticated forecasting models that incorporate a large finite number of lags of inflation, or by responding asymmetrically to average inflation depending on whether inflation is low or high. At the time of this writing, the Fed maintains an opaque AIT strategy, but this need not imply a symmetric view of the target.[5] In fact, Chairman Powell clarified in his announcement of the new regime at the 2020 Jackson Hole Symposium that the Federal Reserve would not be tying their hands "to a particular mathematical formula that defines the average" and that their "approach could be viewed as a flexible form of average inflation targeting" (Powell (2020)). This

[5]It is currently difficult to precisely assess the symmetry of the Federal Reserve's new strategy. Tolerance of some undershooting of the 2% inflation target following the recent surge in inflation may suggest a symmetric approach to AIT.

flexible approach can allow for periodic shortening of the averaging window which helps to avoid unnecessary instability risks.

Our findings furthermore cast doubt on the notion that AIT is a clear compromise between PLT and IT. Under RE, PLT can deliver superior outcomes in terms of social welfare, but there are concerns about the performance of PLT if private sector agents mis-perceive the policy rule and objective. IT regimes, on the other hand, have been implemented successfully and have gained credibility in much of the world. AIT is thought to capture some of the benefits of PLT while staying close to the foundations of IT (e.g. see Nessen and Vestin (2005)). However, we show that AIT can introduce new stability risks that *are not observed under PLT or IT* when agents are learning.

In Section 2 we introduce the AIT formulation, develop the analytical framework and, as reference point, study the transparent case. Sections 3 through 7 present the main stability results about the lack of robustness of AIT under imperfect knowledge and consider various alterations to the AIT framework. Various modelling details and proofs of the results are in the several appendices.

2 New Keynesian Model

This section develops a standard New Keynesian (NK) model of learning. In the model, a continuum of household-firms produce a differentiated consumption good under monopolistic competition and price adjustment costs in the spirit of Rotemberg (1982).[6] Agents optimize over the infinite horizon in accordance with the "anticipated utility" approach formulated by Kreps (1998) and discussed in Sargent (1999) and Cogley and Sargent (2008).[7]

The utility and production functions are assumed to be identical and agents have homogenous point expectations, so that there is a representative agent.[8] Government uses monetary policy, buys a fixed amount of output and finances spending by taxes and issues of public debt. Monetary policy is conducted in terms of an interest rate rule in the cashless limit. It should be recalled that the nonlinear version of the model we use has **two steady states, the inflation target and liquidity trap (or ZLB) steady states**, when interest rate setting follows a suitable nonlinear Taylor rule entailing an active policy response at the target level of inflation. See e.g. Benhabib, Schmitt-Grohe and Uribe (2001) and Benhabib, Evans and Honkapohja (2014) for the RE and learning versions of the model. The existence of two steady states is a key feature also with the PLT and AIT policy rules.

We note that a classical monetary model with full price flexibility is obtained from our model in the limit by setting the adjustment costs to zero. In this case the Phillips curve is replaced by a static first order condition for consumption and labor supply. The distinction between price stickiness and price flexibility turns out to be important for the results and both cases are discussed below. The derivation of the model is given in Benhabib et al.

[6]The Rotemberg model enables study of ZLB regime and global dynamics in the nonlinear system. In contrast, use of Calvo (1983) model of price stickiness, which is the most common NK model, requires linearization. The results of the two models are very similar in the vicinity of the target steady state (see Appendix C.3).

[7]Optimization over infinite horizon is currently the standard assumption in the literature on learning. At the other extreme, one could assume 'Euler equation' learning with agents having one-period ahead horizon. In the current context both approaches give practically the same results for local stability.

[8]Point expectations are an assumption of bounded rationality. It means that agents treat the conditional expectation of a nonlinear function of random variables as equal to the nonlinear function of the conditional expectations.

(2014) and Honkapohja and Mitra (2020). For completeness, formal details of the two versions of the model are derived in Appendix A.

In sections 2 through 4 the linearized model is used to discuss the performance of AIT policy when the economy is near the inflation target steady state and expectations are fairly well anchored. If stability concerns arise in these cases, then we should not expect the policy regime to succeed when inflation is far away from the target (such that the ZLB binds, for example). Section 5 uses the nonlinear model to discuss possibilities of convergence of the economy back to the inflation target when the interest rate is at or very near the ZLB for the interest rate.

2.1 Average Inflation Targeting (AIT)

Formally, it is assumed that the central bank uses an interest rate rule that depends on average inflation and the deviation of output from its target level.

$$R_t = 1 + \max[\bar{R} - 1 + \psi_p[\frac{P_t - \bar{P}_{t,L}}{\bar{P}_{t,L}}] + \psi_y[\frac{y_t - y^*}{y^*}], 0], \tag{1}$$

$$\bar{P}_{t,L} = (\pi^*)^L P_{t-L} \text{ and} \tag{2}$$

$$\pi_t = \frac{P_t}{P_{t-1}}. \tag{3}$$

R_t is the (gross) nominal interest rate, $\bar{P}_{t,L}$ denotes the target price level and π_t is the actual (gross) inflation rate. ψ_p and ψ_y are parameters in the policy rule. It is formulated with a target level for (gross) inflation π^* and $\bar{P}_{t,L}$ is computed by compounding the actual price level L periods ago using target inflation rate π^* of the targeted steady state. y^* is the level of output when inflation is at its target. The rule (1) incorporates the ZLB. Notice that (1) becomes a simple inflation targeting rule when $L = 1$. As $L \to \infty$, (1) becomes a Wicksellian PLT rule with inflation target path given by $\bar{P}_{t,\infty} = (\pi^*)^t P_0$ for all t.[9]

The rule (2) implies that

$$\frac{P_t}{\bar{P}_{t,L}} = \frac{P_t}{P_{t-1}} \cdots \frac{P_{t-(L-1)}}{(\pi^*)^L P_{t-L}} = (\pi^*)^{-L} \prod_{i=0}^{L-1} \pi_{t-i},$$

so the basic AIT rule with the ZLB constraint can be written as

$$\begin{aligned} R_t &= R(y_t, \pi_t, ..., \pi_{t+1-L}) \\ &\equiv 1 + \max\left[\bar{R} - 1 + \psi_p \left[\prod_{i=0}^{L-1} \frac{\pi_{t-i}}{(\pi^*)^L} - 1\right] + \psi_y \left(\frac{y_t}{y^*} - 1\right), 0\right]. \end{aligned} \tag{4}$$

The linearized expression for the interest rate rule (4) near the target steady state is

$$\hat{R}_t = \psi_p \sum_{k=0}^{L-1} \frac{\hat{\pi}_{t-k}}{\pi^*} + \psi_y \frac{\hat{y}_t}{y^*}, \tag{5}$$

where $\hat{y}_t$, $\hat{\pi}_t$ and $\hat{R}_t$ denote deviations from the target steady state, so e.g., $\hat{\pi}_t = \pi_t - \pi^*$.

[9] However, the model properties are not the same for the AIT limit ($L \to \infty$) and PLT ($L = \infty$), as we discuss in section 3.2 and Appendix C.4.

2.2 Behavioral Rules

The NK model with price adjustment costs is often used and its details are well known, so we start directly with the linearized behavioral rules. We use the so-called infinite-horizon framework, where agents focus on expectations over the infinite future.[10] In each time period they make optimal decisions subject to, in general, non-rational expectations obtained from a forecasting model with parameters estimated using past data. Over time the parameters of the forecasting model are revised as new data becomes available.

The analysis relies on two behavioral rules of private agents: the Phillips curve and the consumption function.[11] The linearized Phillips curve takes the form

$$\hat{\pi}_t = \kappa \hat{y}_t + \kappa \sum_{j=1}^{\infty} \beta^j \hat{y}^e_{t+j}, \tag{6}$$

where $\hat{x}$ denotes a linearized variable, and κ is the slope of the Phillips curve and β is the subjective discount rate. Here $\hat{y}_t$ and $\hat{\pi}_t$ denote output and (gross) inflation as deviations from the non-stochastic target steady state. Superscript e indicates expectations while subscripts indicate the periods $t+j$, $j = 0, 1, 2, ...$

The form (6) of the Phillips curve is special as expected future inflation does not directly affect current inflation. (There is an indirect effect via current output in the Phillip's curve.) This formulation is based on the assumption of a representative agent and a simplifying assumption about expectations. One could allow for heterogenous expectations along the learning path. The current formulation facilitates the stability analysis without loss of generality in the results.

The linearized aggregate demand function takes the form

$$\hat{y}_t = -\frac{c^*\beta}{\sigma\pi^*}\hat{R}_t + \sum_{j=1}^{\infty} \beta^j \left(\frac{1-\beta}{\beta} \hat{y}^e_{t+j} - \frac{c^*}{\sigma} \left(\beta \hat{R}^e_{t+j} (\pi^*)^{-1} - \hat{\pi}^e_{t+j} (\beta\pi^*)^{-1} \right) \right). \tag{7}$$

Here c^* and π^* denote the levels of consumption and inflation at the target steady state. σ is the utility function parameter.

There is also a government that consumes amount g_t of the aggregate good, collects the real lump-sum tax Υ_t from each consumer and issues bonds $\mathfrak{b}_t$ to cover financing needs. $\bar{g}$, $\tilde{g}_t$ are the mean and random parts of government spending. Since we consider aspects of stability that are unaffected by the government spending shock, we assume $\tilde{g}_t = 0$ unless otherwise stated throughout the paper. See Appendix A for more details on government behavior. Moreover, it is assumed for simplicity that consumers are Ricardian in the sense that they amalgamate their own intertemporal budget constraint and that of the government (where the latter is evaluated at price expectations of the consumer).

Finally, an "active" or anti-inflationary monetary policy regime is assumed:

[10] There are many different formulations of bounded rationality. One aspect is agents' time horizon: agents might have a limited forecasting horizon. See Honkapohja, Mitra and Evans (2013) for discussion and Hommes, Mavromatis, Ozden and Zhu (2023) for recent empirical applications. Other formulations of expectations formation include natural expectations, see Fuster, Laibson and Mendel (2010) and diagnostic expectations, see Bordalo, Gennaioli and Schleifer (2022). Modeling behavior mechanically or based on optimization is another central aspect of bounded rationality, for recent discussion see e.g. Eichenbaum (2023).

[11] The general description of the nonlinear model and the formal derivation are in Appendix A and its subsections A.1, A.2 and A.3. The linearization is carried out in subsection A.4.

Assumption *The AIT rule satisfies the* ***Taylor principle***

$$\psi_p > \beta^{-1}\pi^* \text{ and } \pi^* \geq 1. \tag{8}$$

(8) is a sufficient condition for (local) determinacy of the target REE under AIT policy.[12] The left-hand side of (8) gives the Taylor rule coefficient for inflation, while the right-hand side is the product of the inverse of real discount factor and gross inflation (i.e. the nominal discount factor) at the target steady state. We impose the Taylor Principle as it is a standard assumption in monetary policy analyses, but we note that it is not a necessary condition for many results in this paper. For example, Propositions 1-3 hold under the weaker restriction: $\psi_p > (L\beta)^{-1}\pi^*$ if $L > 4$.

When the Taylor Principle is satisfied (and $\tilde{g}_t = 0$ as explained above), the unique bounded equilibrium law of motion for $\hat{X}_t = (\hat{y}_t, \hat{\pi}_t, \hat{R}_t)^T$ assumes the minimal state variable (MSV) form:

$$\hat{X}_t = \sum_{j=1}^{L-1} A_j^{RE} \hat{X}_{t-j}. \tag{9}$$

A rational agent would make forecasts consistent with the REE law of motion (9), which requires knowledge of the averaging window (L).

2.3 Stability of the Target Steady State Under Learning

We start by considering as a benchmark the case of "transparency" in which agents know the averaging window L (but not the policy coefficients, ψ_p or ψ_y). The linearized model comprises of the Phillips curve (6), the aggregate demand function (7) and the AIT interest rate rule (5). Expectations, $\{\pi^e_{t+j}, y^e_{t+j}, R^e_{t+j}\}_{j\geq 0}$, are formed under adaptive learning.

Following the literature on adaptive learning, it is assumed that each agent has a model for perceived dynamics of state variables $\{\pi_{t+j}, y_{t+j}, R_{t+j}\}$, also called **the perceived law of motion (PLM)**. The PLM parameters are estimated using available data and the estimated model is used for forecasting. This gives the expectations $\{\pi^e_{t+j}, y^e_{t+j}, R^e_{t+j}\}_{j\geq 0}$ about the future in the current period. The PLM parameters are re-estimated when new data becomes available in the next period. Convergence of the learning dynamics toward the inflation target REE is a fundamental question of interest. For further details see Appendix A, Section A.5.

In linearized models, a common formulation is to postulate that the PLM is a linear regression model, where endogenous variables depend on intercepts, observed exogenous variables and possible lags of endogenous variables. The estimation is based on least squares or related methods. In each period the estimated PLM is used to compute the expectations in the structural model. This yields the temporary equilibrium for the period, also called **the actual law of motion (ALM)**.

Now we formally introduce the general multivariate framework. The general framework is a linearized multivariate model in which agents are forward-looking with infinite horizon and there are $L-1$ lags of endogenous variables. The structural form is

$$\hat{X}_t = K + \sum\nolimits_{i=1}^{\infty} \beta^i M \hat{X}^e_{t,t+i} + \sum\nolimits_{j=1}^{L-1} N_j \hat{X}_{t-j}.$$

[12] Under flexible prices an analytic proof is available and under sticky prices the result holds for the calibrations used. The material is available upon request.

Here $\hat{X}^e_{t,t+i}$ denotes expectations in period t of period $t+i$ endogenous variables. K is a vector of constants. M is the matrix of structural parameters with coefficients given by (6), (7) and (5). The matrices N_j contain the parameters describing the influence of lagged variables of period $t-j$, in particular the parameters of the AIT policy rule (5). Stacking the system into first order form gives the temporary equilibrium system of equations

$$\tilde{X}_t = \tilde{K} + \sum\nolimits_{i=1}^{\infty} \beta^i \tilde{M} \tilde{X}^e_{t,t+i} + \tilde{N}\tilde{X}_{t-1}, \tag{10}$$

where $\tilde{X}_t = (\hat{X}_t^T, \ldots, \hat{X}^T_{t-(L+2)})^T$, $\tilde{X}^e_{t,t+i} = \left((\hat{X}^e_{t,t+i})^T, 0, \ldots, 0\right)^T$,

$$\tilde{M} = \begin{pmatrix} M & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \; \tilde{N} = \begin{pmatrix} N_1 & N_2 & \cdots & N_{L-1} \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & I & 0 \end{pmatrix},$$

$N_1 = ... = N_{L-1} = \hat{N}$ and $\hat{N}$ is given in the proof of Proposition 1 (i). Transparency affords agents the ability to include the correct number of lags of inflation in their PLM.

Definition *Agents are **learning with transparency** if their linear PLM includes exactly L-1 lags of inflation.*

To be concrete, the PLM under transparency is

$$\hat{X}_t = A_0 + \sum\nolimits_{j=1}^{L-1} A_j \hat{X}_{t-j}.$$

The main question of interest is whether agents' PLM will converge to the REE law of motion (9) asymptotically as new data becomes available and agents update their PLM estimates accordingly. Recall that transparency does not imply rationality; learning agents have imperfect knowledge about the economy's structure, including coefficients of the policy rule such as ψ_p, so they estimate the coefficients of their PLM in each period, t, using observable macro data ($\{\hat{X}_k, \tilde{g}_k\}_{k=0}^{t-1}$) and a recursive form of least squares. However, transparency does mean that learning agents know to include the correct variables in their PLM, which may be essential for agents to learn the target equilibrium (cases in which agents include too few lags of inflation (under-parameterization) or too many lags (overparameterization) in their PLM are studied below).

It is useful to express the PLM in vector form

$$\tilde{X}_t = \tilde{A}_0 + \tilde{A}\tilde{X}_{t-1}, \tag{11}$$

where

$$\tilde{A} = \begin{pmatrix} A_1 & A_2 & \cdots & A_{L-1} \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & I & 0 \end{pmatrix} \text{ and } \tilde{A}_0 = \begin{pmatrix} A_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

after substituting the PLM (11) into the ALM (10), the temporary equilibrium induces a mapping PLM→ALM,

$$\begin{aligned} \tilde{A} &\to \sum\nolimits_{i=1}^{\infty} \beta^i \tilde{M} \tilde{A}^{i+1} + \tilde{N}, \\ \tilde{A}_0 &\to \tilde{K} + \sum\nolimits_{i=1}^{\infty} \beta^i \tilde{M}(I + \tilde{A} + ... + \tilde{A}^i)\tilde{A}_0, \end{aligned} \tag{12}$$

denoted as $T(\tilde{A}_0, \tilde{A})$, in the space of PLM parameters. Notice that the REE coefficients in (9) are a fixed point of this mapping. Further, this mapping yields a differential equation

$$\frac{d(\tilde{A}_0, \tilde{A})}{d\tau} = T(\tilde{A}_0, \tilde{A}) - (\tilde{A}_0, \tilde{A}), \tag{13}$$

where τ is virtual time. Local stability conditions of (13) at the fixed point characterized by (9) yield the E-stability conditions which are usually the key conditions for convergence of real-time adaptive learning to the REE of interest.[13] The conditions are formally given in Appendix A.6, see equations (63).

For brevity, we only check that these stability conditions are satisfied in a calibrated version of the model for the case of learning with transparency. It should be emphasized that at times our analysis must use calibrations of the model as explicit economically interpretable theoretical conditions cannot always be obtained. The calibrations used throughout the paper are taken from the literature on New Keynesian models, and our objective is only to illustrate the likely implications for the general model. As such, we do not check that our calibrated models have strong empirical support. In the literature suggested calibrations for price adjustment costs, γ, which affect the Phillips curve slope κ (i.e., $\partial\kappa/\partial\gamma < 0$ and $lim_{\gamma\to 0}\kappa = +\infty$), vary a great deal as they depend on estimates of frequency of price adjustment and markup and there are different estimates for both. For recent discussion see Honkapohja and Mitra (2020) who use the alternative values $\gamma = 42$, 128.21 or 350 for the price adjustment cost parameter, consistent with empirical evidence. For conciseness, only a single standard calibration is adopted for other parameters of our quarterly framework: $\pi^* = 1.005$, $\beta = 0.99$, $\alpha = 0.7$, $\nu = 21$, $\sigma = \varepsilon = 1$, and $g = 0.2$. Policy parameters for the AIT rule are set at $\psi_p = 1.5$, $\psi_y = 0.125$.

For the case of transparent AIT, our numerical E-stability results are summarized as follows:

Remark 1: *Assume that the policy rule (5) satisfies the Taylor principle (8) and agents are learning with transparency. The target REE is E-stable under fully flexible or sticky prices if L is not too large, but is not E-stable for sufficiently large L when prices are sticky.*

Remark 1 highlights that transparency can anchor agents' expectations. However, convergence of learning is not likely to occur if the window length (L) is too large. We find that when prices are relatively flexible ($\gamma = 42$), the REE is E-unstable for $L > 36$ (9 year window), whereas the REE is E-unstable for $L > 22$ (5.5 year window) when prices are relatively rigid ($\gamma = 128.21$ and $\gamma = 350$). Thus, there are risks to instability when the central bank is transparent about the window, the window is too long, and a unique bounded REE exists. However, plausible calibrations of the model indicate that AIT can guide the economy to the target equilibrium if the averaging window is fairly short.

3 Imperfect Structural Knowledge and Learning

Recalling that the Federal Reserve has given little information about the details of AIT, we now start to consider situations where the policy structure, including the averaging length, L, is not known to the agents. Under this opacity about the monetary policy framework, agents try to forecast the interest rate as well as output and inflation rate without any

[13] Appendix A.5 gives more detailed discussion of real-time algorithms and E-stability.

knowledge of the number of inflation lags in the policy rule (e.g. $L-1$). Some of the references cited in the introduction may suggest that agents are likely to exclude some crucial inflation lags from their PLM.

As a first case, it is assumed that agents exclude lagged endogenous variables from their PLM ("full opacity"). Below we consider more general cases of "opacity" in which agents' PLM incorporates some but fewer than $L-1$ number of lags. The stability properties are the same for these intermediate cases. The next section analyzes the arguably less likely case that agents' PLM has more inflation lags than the policy rule.

Definition *Agents are* ***learning with full opacity*** *if they exclude all lagged endogenous variables from their linear PLM, but possibly include other observables.*

Note that the only lags in the model are lagged inflation rates in the policy rule and private agents have no knowledge of the form (4). Consequently, it is not implausible to think they would exclude these variables from their forecasting models. Also note that agents' PLM under full opacity can be the model-consistent PLM in the model with a simple IT rule ($L=1$), and therefore under full opacity agents forecast as if they are living under a standard IT regime. With AIT under full opacity, the equilibrium involves an under-parameterized forecasting model and thus the possible long-term outcome is a **restricted perceptions equilibrium**.[14]

Our interest is the stability of the model's target steady state, which can be validly assessed under the simplifying assumption that the random part of government spending $\tilde{g}_t$ is identically zero (see Appendix A.5 for more details). This assumption distills the full opacity PLM down to an intercept term (i.e. agents estimate the long-run mean values of state variables). We assume agents estimate these long-run mean values using a steady state learning scheme which is formalized as

$$s^e_{t+j} = s^e_t \text{ for all } j \geq 1, \text{ and } s^e_t = s^e_{t-1} + \omega_t(s_{t-1} - s^e_{t-1}), \tag{14}$$

where $s = \hat{y}$, $\hat{\pi}$, $\hat{R}$. In this notation expectations s^e_t refer to future periods (and not the current one) formed in period t. When forming s^e_t the newest available data point is s_{t-1}, i.e. expectations are formed in the beginning of the current period.

In adaptive learning the gain parameter ω_t is usually chosen to exhibit either decreasing gain, with the sequence $\{\omega_t\}$ converging to zero, or constant gain, with the gain parameter as a small constant $\omega_t = \omega \in (0,1]$. The E-stability concept used in Remark 1 above involves 'Decreasing gain' learning. Below 'Constant gain' learning is mostly used.[15]The latter concept is commonly used in applied studies and is necessary if the speed of learning is an object of interest. See Appendix A.5 for further remarks on learning.

3.1 Temporary Equilibrium and Full Opacity

Under steady state learning with full opacity, agents form expectations $\hat{\pi}^e_{t+j} = \hat{\pi}^e_t$, $\hat{y}^e_{t+j} = \hat{y}^e_t$ and $\hat{R}^e_{t+j} = \hat{R}^e_t$ for $j = 1, 2, ...$ according to (14) at the beginning of time t. Using the equations (6) and (7) with the form of expectations just given, time-t temporary equilibrium

[14]See e.g. Evans and Honkapohja (2001) and Branch (2006). The term self-confirming equilibrium is also used in in the literature, see e.g. Sargent (1999).

[15]The two notions of stability are closely related, as is evident from the Definition in Section 3.2. With decreasing gain the parameters converge to a fixed point, whereas with constant gain, convergence is to a random variable with distribution concentrated around its long-run mean, see Section 7.4 of Evans and Honkapohja (2001).

with steady state learning is determined by:
(i) the infinite horizon Phillips curve

$$\hat{\pi}_t = \kappa \hat{y}_t + \frac{\kappa\beta}{1-\beta}\hat{y}_t^e, \tag{15}$$

(ii) the aggregate demand function coupled with market clearing,

$$\hat{y}_t = -\frac{c^*\beta}{\sigma\pi^*}\hat{R}_t + \hat{y}_t^e - \frac{c^*\beta}{\sigma\pi^*}\left(\frac{\beta}{1-\beta}\hat{R}_t^e - \frac{\beta^{-1}}{1-\beta}\hat{\pi}_t^e\right) \tag{16}$$

(iii) and the linearized interest rate rule (5). See Appendices A.2 and A.3 for derivation of (15) and (16).

The system is compactly written

$$\begin{aligned} \hat{y}_t - Y(\hat{y}_t^e, \hat{\pi}_t^e, \hat{R}_t, \hat{R}_t^e) &= 0, \\ \hat{\pi}_t - \Pi(\hat{y}_t, \hat{y}_t^e) &= 0, \\ \hat{R}_t - R(\hat{y}_t, \hat{\pi}_t, ..., \hat{\pi}_{t+1-L}) &= 0, \end{aligned}$$

or,

$$F(\hat{X}_t, \hat{X}_t^e, \hat{X}_{t-1}, ..., \hat{X}_{t-(L-1)}) = 0, \tag{17}$$

where F consists of the aggregate demand function, the Phillips curve and the interest rate rule. The vector of current state variables is $\hat{X}_t = (\hat{y}_t, \hat{\pi}_t, \hat{R}_t)^T$ while $(\hat{X}_{t-1}, ..., \hat{X}_{t+1-L})^T$ contains the lagged endogenous variables. Because the model under study is linearized, F and the law-of-motion for expectations under learning with full opacity is expressed as:

$$\begin{aligned} \hat{X}_t &= (1-\omega)M\hat{X}_{t-1}^e + (\omega M + N_1)\hat{X}_{t-1} + \sum_{i=2}^{L-1} N_i \hat{X}_{t-i} & (18) \\ \hat{X}_t^e &= (1-\omega)\hat{X}_{t-1}^e + \omega\hat{X}_{t-1}, \text{ where} & (19) \end{aligned}$$

Here $\hat{X}_t = (\hat{y}_t, \hat{\pi}_t, \hat{R}_t)^T$ with the hat denoting a linearized variable and the matrices M, N_1, ..., N_{L-1} are given in the Appendix D. 1. Recall also that $\hat{X}_t^e$ refers to expected future values of $\hat{X}_t$ and not the current one.

3.2 Stability and Full Opacity

We now consider if AIT can guide actual and expected inflation to the desired target level when inflation and expected inflation are initially (arbitrarily) close to the target levels. To gauge whether the central bank's target equilibrium is "expectationally stable" or "locally stable" in this sense, we focus on "small gain" results, i.e. whether stability obtains for all ω sufficiently close to zero.

Definition *The steady state is said to be **expectationally stable** or **(locally) stable under learning** if it is a locally stable fixed point of the system (18) - (19) for all* $0 \leq \omega < \bar{\omega}$ *for some* $\bar{\omega} > 0$.

Conditions for this can be directly obtained by analyzing (18)-(19) in a standard way as a system of linear difference equations.[16] Intuitively, local instability means that an

[16] Alternatively, so-called expectational stability (E-stability) techniques based on an associated differential equation in virtual time can be applied, as we have already done in section 2.3. E.g. see Evans and Honkapohja (2001).

arbitrarily small disturbance to agents' expectations causes the economy to permanently diverge away from the steady state. Local instability is, therefore, a serious warning signal about the performance of monetary policy. We find there is local stability of constant gain learning with full opacity if there is price stickiness and the Taylor Principle is satisfied. In the case of full price flexibility, local instability obtains under learning even if the Taylor principle holds.

Proposition 1 *Assume that the Taylor principle (8) holds for the policy rule (5).*
(i) Assume that there is price stickiness ($\infty > \kappa > 0$). Then for small ω, the target steady state is locally stable under constant gain learning with full opacity for all L.
(ii) Assume that there is full price flexibility ($\kappa \to \infty$). For small ω, the target steady state is locally stable under constant gain learning with full opacity for $L \leq 3$ but is unstable for higher values of L.[17]

Proofs are given in the Appendices D.1 and D.2. As noted in the proof, the Taylor Principle is not necessary for the result if $L > 4$. Proposition 1 (i) and (ii) raise questions about applicability of the results. As stability is overturned in the limit $\kappa \to \infty$ to price flexibility, it is imperative to study whether the AIT rule ensures a stable equilibrium for empirically plausible values of the gain parameter when there are positive adjustment costs $0 < \kappa < \infty$.[18] There is no unambiguously accepted range for the gain parameter but the range could be something like $[0.002, 0.04]$,[19] and so we propose the following relatively conservative definition of robust stability.

Definition *The steady state is said to be* ***robustly stable under learning*** *if it is a locally stable fixed point of the system (18) and (19) for all* $0 < \omega \leq 0.01$.

We need to calibrate the system (18) and (19) in order to assess robust stability when $0 < \kappa < \infty$. To this end, we use the calibrations reported earlier. For the three calibrations of γ we compute the (approximate) least upper bound for the gain parameter ω_0, so that values $\omega > \omega_0$ lead to instability of the target steady state in the calibrated model. In Appendix C.2, we repeat this analysis for other values of the price and output reaction coefficients in the AIT rule (4), and for a case in which the Phillips curve is especially flat (κ is very small). The basic result is:

Remark 2 *Under full opacity the calibrated model with sticky prices is not robustly stable for higher values of L. Robustness, as measured by ω_0, diminishes with increased window length L.*

γ	42	128.21	350
ω_0 (IT)	0.04242	0.04545	0.05316
ω_0 (PLT)	0.00909	0.00582	0.00336
ω_0 (AIT with $L = 6$)	0.00281	0.00417	0.00513
ω_0 (AIT with $L = 20$)	0.00022	0.00047	0.00087
ω_0 (AIT with $L = 32$)	0.00008	0.00019	0.00038

[17] In contrast to Proposition 1(ii), Honkapohja and Mitra (2020) find that the target steady state is locally stable under learning with opacity and PLT in the flexible price version of the model studied here.

[18] The idea of using the range of values for gain parameter as a criterion for robustness was first suggested in Evans and Honkapohja (2009b).

[19] See e.g. Orphanides and Williams (2005), Branch and Evans (2006), Milani (2007) and Eusepi, Giannoni and Preston (2018).

Table I: Least upper bounds in AIT, IT and PLT

Table I demonstrates that while the IT rule is robustly stable, the AIT and PLT rules are not robustly stable. The latter two are fairly similar in terms of robustness if L is not too high (e.g. $L = 6$ is in the range of optimal averaging windows in Amano et al. (2020)). However, for higher values of L such as $L = 20$ or $L = 32$, which correspond to 5-year and 8-year averages, respectively, we observe instability even for values of ω that are implausibly low. Finally, to the extent that price rigidity appears to stabilize expectations, AIT might appear to function better when estimates of the Phillips curve slope are fairly small. However, this is a bad sign for AIT if high inflation tends to be associated with a steeper Phillips curve when inflation is away from the ZLB.

Table 1 also reveals an important discontinuity: stability outcomes deteriorate as L increases, but stability concerns are more benign when the averaging window is infinite (i.e., under PLT). In fact, PLT is not susceptible to the same issues we have observed thus far under AIT. First, communication about PLT can dramatically improve stability outcomes – even if agents do not know the functional form of the interest rate rule – as documented under analogous conditions of learning by Honkapohja and Mitra (2020).[20] On the other hand, transparency about AIT may not be consistent with expectational stability if L is finite and long, as discussed earlier. Second, AIT under full opacity leads to instability for any value of the gain parameter when prices are fully flexible, whereas the target equilibrium remains stable under PLT with full opacity when prices are fully flexible as shown by Honkapohja and Mitra (2020).

The instability problem under AIT reflects a tension between stabilizing a *finite* moving average of inflation and stabilizing long-run inflation expectations which are influenced by data. A finite window lets bygones be bygones; the policymaker aims for above-target inflation following a period of low inflation, but then it eventually drops the old low inflation data from their measure of average inflation and aims to undershoot the target to compensate for the overshooting. In the model with flexible prices, full opacity and gain equal to zero ($\omega = 0$), this pattern of under- and over-shooting as bygones become bygones implies a deterministic cycle for equilibrium inflation: $\pi_t = \pi_{t-L}$.[21] Thus, in the absence of learning ($\omega = 0$), inflation never returns to target following an initial shock. Adding learning with opacity ($\omega > 0$) and flexible prices only makes the issue worse; agents who observe the pattern of over- and under-shooting end up confusing some part of the transitory rise in inflation for higher permanent inflation and raise long-run expectations accordingly.

This de-anchoring of expectations from the target exacerbates the inflation overshooting of the target relative to the case of rational agents, and leads to oscillations in temporary equilibrium inflation that diverge to positive or negative infinity. Price stickiness helps to stabilize the economy, but does not eliminate the oscillations in inflation, and for given $\omega > 0$, instability is more likely to obtain if L is very large. Under PLT, on the other hand, bygones are not bygones and consequently the pattern of over- and under- shooting is not observed under PLT with full opacity and small gain. In the special case of no learning with full opacity ($\omega = 0$), equilibrium inflation converges *monotonically* to the target following an initial shock to the economy. Increasing the gain moderately under PLT does not give rise to oscillations that are difficult for agents with opacity to correctly forecast.

[20]It may be recalled that according to Honkapohja and Mitra (2020) performance of PLT is much improved if private agents use more information about the policy framework.

[21]See Appendix C.4 for derivations in this paragraph.

Our instability concern is not limited to the model with Rotemberg pricing frictions. The result in Proposition 1 applies to a model with Calvo pricing friction. Similar robust stability concerns emerge in a calibrated version of that model for the case of AIT with large L (and robust stability concerns are somewhat mitigated under PLT–see Appendix C.3).

Remark 3: *The implications of AIT policy (5) in the Calvo NK model, e.g. Preston (2005), are very similar to those in Proposition 1.*

Looking at the results so far it is apparent that a fully opaque average inflation targeting framework can pose serious risks to economic stability. In principle, communication about the averaging window may mitigate concerns about robust stability, as a transparent AIT framework with moderately short averaging window, discussed in Section 2.3, does not pose the same stability risks.[22] Similar concerns emerge if agents include some but not all of the relevant lags of inflation in their forecasting models, as we show next.

3.3 Opacity: AR(1) case

A natural alternative to the full opacity PLM (which excludes all lags of inflation) is the following PLM:

$$s^e_{t+j} = a_{s,t} + b_{s,t}\hat{\pi}^e_{t+j-1}, \tag{20}$$

where $s = \hat{y}, \hat{\pi}, \hat{R}$. The PLM (20) is similar to PLMs studied in Hommes and Zhu (2014), and it encodes agents' belief that inflation is serially correlated. As noted previously, we say that agents have "opacity" but not "full opacity" if they include one lag of inflation in their PLM. We start with the case in which agents include only a single lag of inflation in their PLM, and with some abuse of terminology, we refer to this is as the "$AR(1)$" case. The next section considers other underparameterized forecasting models with more than one but less than $L-1$ lags.

Definition *Agents are **learning with opacity** if they include at least one but less than $L-1$ lags of inflation in their linear PLM.*

As with full opacity, the PLM (20) is under-parameterized relative to the MSV REE law of motion, and therefore agents cannot learn an REE under opacity. However, agents' beliefs may nonetheless converge to a self-confirming restricted perceptions equilibrium (RPE), (a_s, b_s), where (a_s, b_s) satisfy the following least squares orthogonality restriction:

$$E\hat{\pi}_{t-1}\left(s_t - a_s - b_s\hat{\pi}_{t-1}\right) = E(\pi_{t-1} - a_s)\left(s_t - a_s - b_s\hat{\pi}_{t-1}\right) = 0, \tag{21}$$

if such an RPE exists. Hommes and Zhu (2014) and Hommes et al. (2023) use a similar orthogonality restriction to identify self-confirming equilibria of univariate and multivariate models with $AR(1)$ PLMs. Intuitively, (21) ensures that agents' have self-confirming beliefs about the mean and the 1st auto-correlation in the data, and hence cannot easily detect the fact that their PLM is misspecified.

We now consider stability of learning in the $AR(1)$ case of opacity. To examine the possibility of stability in a parsimonious fashion, we assume there exists a b_s which satisfies

[22] Appendix C.2 demonstrates that the calibrated model is robustly stable under learning with transparency for shorter averaging windows. When conducting the robust stability analysis under transparency, we first verify E-stability and then fix the estimates of the AR coefficients in the PLM to their RE values and let agents estimate the intercept term in their PLM only.

(21), and that agents set $b_{s,t} = b_s$ and estimate $a_{s,t}$ using the following constant-gain learning algorithm:[23]

$$a_{s,t} = a_{s,t-1} + \omega(s_{t-1} - b_s\hat{\pi}_{t-2} - a_{s,t-1}). \tag{22}$$

Expectations, s^e_{t+j}, are formed by substituting (22) and $b_{s,t} = b_s$ into (20), and temporary equilibrium is given by (5)-(7). In this setting, we say that the target steady state is stable under learning with opacity if agents learn an RPE near the target (i.e. an RPE for which $a_s = 0$ for $s = \hat{y}, \hat{\pi}, \hat{R}$). As it turns out, the target steady state is stable under learning if prices are sufficiently sticky, but is unstable under learning if prices are flexible.

Proposition 2 *Assume that the Taylor principle (8) holds for the policy rule (5).*
(i) Assume that there is price stickiness ($\infty > \kappa > 0$). For small ω and high γ, the target steady state is locally stable under constant gain learning with AR(1) PLM (20) for all L.
(ii) Assume that there is full price flexibility ($\kappa \to \infty$). For small ω, the target steady state is locally stable under constant gain learning with AR(1) PLM (20) for $L \leq 3$ but is unstable for many higher values of L.

Comparing Proposition 1(i) (Proposition 1(ii)) to Proposition 2(i) (Proposition 2(ii)) it is evident that AIT poses similar risks to stability under both opacity and full opacity. Badly under-parameterized PLMs, which could arise due to a lack of transparency about the policy framework, or because households do not systematically condition their inflation expectations on a sufficiently large number of lags of the quarterly inflation rate, can undermine the efficacy of AIT. As in Proposition 1, the Taylor Principle is not a necessary condition for the result in Proposition 2 (see Appendix D).

Of course there are many other under-parameterized PLMs one may consider, particularly if L is large. In the next section we provide results for these cases.

3.4 Other Underparameterized Lag Structures

It is now assumed that the lag structure in the PLM has $N-1$ lags on inflation , where $3 \leq N < L$. Temporary equilibrium is as before, see (10). In this setup the mapping from PLM to ALM has to incorporate the feature that the PLM is a $VAR(N-1)$ process while the resulting ALM is $VAR(L-1)$. It is necessary to project the ALM into the subspace of $VAR(N-1)$ processes to obtain the best linear predictor in this class. The RPE is a fixed point of the map from the PLM to the projected ALM. The details are in Appendix B.

In the sticky price setup it is not possible to obtain analytical results in the cases $1 < N-1 < L-1$, so we revert to numerical analysis. We use the basic calibration described above. According to the numerical results, the stability result of the $AR(1)$ case in Proposition 2 (i) generalizes in the sticky price economy:

Remark 4 *The result of Proposition 2 (i) continues to hold in the sticky price economy with underparameterized learning.*

Table II illustrates robustness of stability in intercept learning for $L = 8$, when agents use the RPE values for AR parameters in the PLM. (Appendix B provides the technical details for the computation.) Table II gives (approximate) the least upper bounds ω_0 for stability.

[23] Note that we assume that $\tilde{g}_t$ is a small i.i.d shock to ensure a well-defined b_s, and it can be shown that $a_s = 0$ (i.e. the target steady state is a fixed point of the system).

$N-1$	0 (Full Op.)	1	2	3	4	5	6	$L-1$
ω_0	0.00258	0.00268	0.00327	0.00432	0.00631	0.01024	0.01979	0.02383

Table II: Least upper bound for stability of the gain parameter in underparameterized constant gain learning ($L = 8$).

In the flexible price case theoretical results can be obtained and the outcome is analogous to Proposition 2 part (ii), so the RPE is unstable under learning when agents' PLM is underparameterized.

Proposition 3 *Assume that the Taylor principle (8) holds for the policy rule (5). In the flexible price economy with underparameterized learning there is no stationary learning-stable RPE when* $N = 3, ..., L-1$.

The proof of Proposition 3 is in Appendix D.

Non-existence of a stationary RPE makes it difficult to numerically compute a good approximation to the RPE. If the RPE values of the PLM parameters are not known, then it can be hard to find initial conditions for a learning process that are in a small neighborhood of the RPE. Appendix B contains further discussion.

4 Can Agents Learn the Length of the Data Window of AIT Rule?

The preceding section showed how difficulties with stability arise if the data window length N in agents' PLM is shorter than the window length L of the policy rule. In the opposite case, where agents' data window has length $R > L$, it turns out that, given other standard assumptions of the policy rule parameters, private agents can learn the value of L if they start with the PLM with a longer data window than the window in the AIT rule.

The general form of the model is

$$\begin{aligned} \tilde{X}_{R,t} &= (\hat{X}_t^T, ..., \hat{X}_{t-(R-2)}^T)^T \\ \hat{X}_t &= K + \sum_{i=1}^{\infty} \beta^i M \hat{X}_{t,t+i}^e + \sum_{j=1}^{L-1} N_j \hat{X}_{t-j}. \end{aligned} \tag{23}$$

As before, the lagged term $\sum_{j=1}^{L-1} N_j \hat{X}_{t-j}$ incorporates lags of endogenous variables which in the current model consist of lagged inflation rates of the AIT rule. The PLM allows for overparameterization ($R > L$)

$$\hat{X}_t = A_0 + \sum_{j=1}^{R-1} A_j \hat{X}_{t-j}. \tag{24}$$

The model with overparameterization is analyzed in detail in Appendix D.5. It is shown there that the MSV solution must fulfill the following lemma:

Lemma 4 *For any MSV REE solution of the model (23) the PLM (24) satisfies the equations*

$$A_L = ... = A_{R-1} = 0$$

in both sticky-price and flexible-price models.

In other words, in the REE the overparameterized PLM has zero coefficients in the lags that are not included in the correctly specified PLM. The proof is in Appendix D.5.

It should be noted that the REE with overparameterized PLM may not be unique as the equations for determining the values of the PLM parameters are nonlinear and can have multiple solutions.[24] To proceed, a selection criterion among the MSV solutions must be used and, as before, we select the MSV solution that is locally determinate in the further analysis. Convergence of learning to MSV solution is then considered by requiring E-stability of the solution of interest.

In this kind of situation it is useful to make the distinction between weak and strong E-stability, see Evans and Honkapohja (2001), Chapter 9, sections 3-5. **Weak E-stability** refers to E-stability of REE when agents' PLM has the same functional form as the REE of interest. **Strong E-stability** refers to E-stability of REE when agents' PLM is overparameterized relative to the REE of interest. Using simple models, Evans and Honkapohja (2001) show that conditions for strong E-stability add a further requirement to the weak E-stability conditions.

At this stage it is necessary to revert to numerical analysis as the equations for determining the MSV solution of interest and the analysis of its stability under learning do not yield analytically interpretable forms of stability conditions. The numerical analysis employs the benchmark calibrations described earlier. Figure IA (top panel) uses the standard calibration also for the policy parameters $\psi_p = 1.5$ and $\psi_y = 0.125$ and uses the grid $L \in [0, 40]$ for the averaging window of the policy rule and the grid for the PLM window $R = L + Q$ with $Q \in [0, 10]$. The shaded area indicates strong E-stability. It is seen that there are differences in stability areas for different values of the price stickiness parameter γ. Figure IB (bottom panel) uses a policy rule with $\psi_p = 2$ and $\psi_y = 0.125$, so the rule has a more aggressive response with respect to inflation.

FIGURES IA AND IB ABOUT HERE

The figures illustrate the following result, which is a consequence of Lemma 4:

Proposition 5 *Consider the NK model with policy rule (5) and assume that the Taylor principle (8) holds at a locally determinate MSV solution. Then weak E-stability of the MSV solution is a necessary condition for the solution to be locally E-stable with overparameterized PLM (strong E-stability). The same result holds in the flexible price case.*

In an earlier section, we showed that the REE is weakly E-unstable if L is too large. Proposition 5 therefore informs us that agents cannot learn the window or REE with transparency or over-parameterized PLM if the true averaging window is too long. However, agents can learn the window length and target REE with over-parameterized PLM, provided that L is sufficiently small and the parameters of the rule are set so that the target equilibrium is (locally) determinate and strongly E-stable:

Corollary 6 *If the MSV REE of the economy is strongly E-stable, private agents can learn the correct data window length in the policy rule if they have more inflation lags in their PLM than the number of lags in the AIT policy rule.*

[24] Evans and Honkapohja (1992) and Chapter 9 of Evans and Honkapohja (2001) discuss the set of overparameterized solutions to linear scalar RE models with lagged endogenous variables.

Suppose the central bank targets a simple moving average of inflation as in the benchmark analysis and the economy is strongly E-stable. If the central bank does not communicate L, agents can try to learn the averaging window by estimating a $VAR(R-1)$ model for inflation, output and the interest rate. If $R < L$ then agents will not learn the true window, and expectations will likely diverge from the target as shown earlier. However, if $R \geq L$, agents will learn the window if the estimates of the coefficients on extraneous lags of the endogenous variables go to zero asymptotically.

Figures IA and IB show numerically that agents may learn the window, provided $R \geq L$. The MSV solution is strongly E-stable whenever it is weakly E-stable. However, it is also shown that the MSV solution may not be E-stable if the averaging window is too long. Hence, lengthening the averaging window can pose risks to instability when agents forecast in a sophisticated manner and try to learn the averaging window by estimating an overparameterized PLM recursively. The risk of instability under long averaging windows therefore emerges under both opacity and the MSV/overparameterized PLM cases.

This above approach mimics the "general-to-specific t rule" for the choice of lag length in time series analysis of AR processes. There are also more sophisticated approaches to selecting the number of lags of a VAR model: use of AIC or BIC information criteria for model selection would be another approach. See e.g. Hayashi (2000), Section 6.4 for these rules and criteria. For reasons of space we refrain from the analysis of the latter criteria.

5 Escape From the ZLB Regime With AIT?

One key argument for introducing AIT policy in place of IT has been its potential in providing a framework that facilitates return from the regime of very low interest rates to a normal regime with the economy operating near the inflation target equilibrium. From the outset, we noted that AIT under full opacity is not a robust mechanism for escape from the ZLB; even if the economy escapes the ZLB regime under AIT with full opacity, the instability or non-robust stability of the target equilibrium implies that inflation may never converge to the target. Further, when the ZLB is binding, the dynamics under AIT with full opacity are identical to the dynamics under IT, and so the results from earlier analyses which cast doubt on the efficacy of IT at the ZLB can be applied.[25] On the other hand, the target steady state is more stable under learning with transparency. Can a transparent AIT rule bring the economy back to the target steady state from the ZLB regime?

We consider the issue of escaping the ZLB under AIT by using a stochastic version of the nonlinear model with (4) and agents' PLM taking a form similar to (11)[26]

$$\tilde{X}_t = \tilde{A}_t \tilde{X}_{t-1} + \tilde{A}_{0,t} + \tilde{B}_t \tilde{g}_t.$$

For this section, agents are assumed to learn about all parameters in their learning rule. At the end of each period t, agents update their estimates, $\xi_t = (\tilde{A}_{0,t}, \tilde{A}_t, \tilde{B}_t)^T$, using all data available at the end of the period using a standard constant gain least-squares algorithm:

$$\begin{aligned} \xi_t &= \xi_{t-1} + \omega S_t^{-1} z_t \left(\tilde{X}_t - \xi_{t-1}^T z_t \right)^T, \\ S_t &= S_{t-1} + \omega (z_t z_t' - S_{t-1}), \end{aligned}$$

[25]See Evans, Guse and Honkapohja (2008) and Benhabib et al. (2014).

[26]In these simulations we include a small shock (i.e. $\tilde{g}_t \neq 0$) to mitigate multicollinearity issues which arise when agents jointly estimate the intercept and lagged variable coefficients in the non-stochastic version of the model. See e.g. Evans and Honkapohja (2001), Chapter 7 for a review of related issues.

where $z_t = (1, \tilde{X}_{t-1}^T, \tilde{g}_t)^T$.[27] Agents make forecasts using $\tilde{A}_{0,t}, \tilde{A}_t, \tilde{B}_t$ in period $t+1$. Agents are assumed to understand that they live in the nonlinear model, and so $x_{t+j}^e = exp(\hat{x}_{t,t+j}^e)$, where $x = y, R, \pi$, hatted variables are the logs of variables and also the elements of stacked vector, $\tilde{X}_t$, and x_{t+j}^e is the time-t expectation of x_{t+j}.

As a first example consider the case where the economy is initially very near the low steady state (y_{Low}, π_{Low}) with binding ZLB, such that $\pi_0^e = \pi(0) = \pi(-1) = \ldots = \pi(-L+1) \approx \pi_{Low}$, $y_0^e = y_0 \approx y_{Low}$, $R_0 = R_0^e \approx 1$, where y_0^e, π_0^e, and R_0^e denote the initial expected *long-run* levels of y, π, and R, respectively.[28] We assume $\tilde{A}_0$ is a zero matrix and $\tilde{A}_{0,0}$ is set in accordance with R_0^e, π_0^e, and y_0^e. Our assumption about $\tilde{A}_0$ may be a reasonable description of beliefs at the beginning of a transition from a standard IT policy regime to a well-communicated AIT regime. The basic calibration is the same as earlier in Section 3 with $\gamma = 128.21$, $\omega = 0.005$, and $L = 6$.[29] The economy escapes the liquidity trap in this case, as shown by the blue curves in Figure II.

Figure II A-C HERE

AIT with learning under transparency is also compared with the IT policy framework in Figure II. From the figure, we see that AIT generates makeup inflation and brings inflation to the level of the target much faster than under IT, though the makeup inflation comes at the expense of greater output volatility. In both cases, the economy converges to the target steady state. However, this finding is not robust; when we vary L and ω, and repeat the same simulation from the low steady state, we observe convergence to target under IT for much higher values of the gain parameter than under a transparent AIT regime (see Table III). Deflationary spirals take place in simulations that do not converge to the target steady state.[30] Table III also gives an example of non-robust stability of learning in the case of transparent AIT.

L	1 (IT)	4	6	12	20
ω_0	0.05629	0.01774	0.01138	0.00584	0.00362

Table III: Least upper bounds ω_0 for stability

Whether AIT under transparency initiates escape from the liquidity trap depends also on assumptions about initial expectations and economic conditions. The **domain of escape**[31] from the liquidity trap for different initial conditions $\pi_0^e \approx \pi(0) = \pi(-1) = \ldots = \pi(-L+1)$, $y_0^e \approx y_0$, and $R_0 = R_0^e \approx 1$, with $L = 6$ and low gain parameter, $\omega = .002$, is shown below in Figure III.

[27] In simulations, agents are assumed to know that lags of output and the interest rate do not matter in equilibrium. Suitable modifications to the learning algorithm are made to impose this assumption. We assume S_{-1} is proportional to the identity matrix in numerical simulations.

[28] In fact to facilitate the numerics we set R_0 and R_0^e slightly above 1.

[29] Earlier sections and Appendix C.2 analysis indicates that lower values of L may lead to better stability outcomes than higher values. We choose a value of L that is in the range of optimal averaging windows studied by Amano et al. (2020).

[30] For the numerical Table III simulations, we conclude that the economy diverges from the target if the the economy does not return to the steady state within 50,000 periods.

[31] Domain of escape from the liquidity trap is the set of initial conditions near the low steady state that lead to convergence to target steady state. It is part of the domain of attraction of (π^*, y^*).

Figure III HERE

It is seen that there is a domain of escape from the liquidity trap, but it covers only a small area around the low steady state. In particular, if y_0^e is below y_{Low} and π_0^e is approximately at level π_{Low}, the economy does not escape from the liquidity trap.[32] By this measure both AIT and IT are less robust than PLT under similar information settings (see Honkapohja and Mitra (2020) for the corresponding results under PLT, and Appendix C.1 for the domain of escape under IT).

We did not find the basic results in this section to be sensitive to initial values of the lag coefficients of the PLM parameters. Figure A.2 in the Appendix presents the domain of escape when agents' initial beliefs about the lag coefficients, $\tilde{A}$, coincide with the coefficients of the unique, stable MSV solution of the linearized model (i.e. we impose $\tilde{A}_0 = \bar{A}_0$, where $\bar{A}_0$ is from the model linearized around the target steady state). Under this assumption about initial beliefs, agents understand that the policymaker aims for makeup inflation following a ZLB event, and yet still the performance of AIT is not significantly improved.

Our analysis shows that the performance of AIT policy in the nonlinear model with the ZLB is sensitive to the speed of learning, just as the success of AIT under transparency near the target steady state hinges on the magnitude of the gain parameter. Further, AIT does not clearly outperform IT when expectations are near the low steady state–even if agents understand the basic structure of AIT and the implication of make-up inflation–and it very clearly underperforms a credible PLT regime (see Honkapohja and Mitra (2020)).

6 Asymmetric AIT

The previous sections focus on rules that respond symmetrically to positive and negative deviations of average inflation from the target. It was shown that the central bank may have to commit to a credible, transparent *and sufficiently small* averaging window, or risk losing control of inflation under a symmetric rule. Such commitment may be undesirable to the policymaker depending on their preferences. For example, opacity about the averaging window can make it difficult for private sector agents to attribute a persistent gap between current inflation and the target level to monetary policy mistakes. Moreover, the assumption of symmetry might ultimately prove incompatible with the Federal Reserve's evolving strategy. To overcome these concerns, alternative asymmetric rules have been proposed in the literature.[33] We briefly consider performance of the following asymmetric AIT rule:

$$R_t = 1 + \max[\bar{R} - 1 + \psi_p[\mathcal{P}_t - 1] + \psi_y[\frac{y_t - y^*}{y^*}], 0], \text{ where} \tag{25}$$

$$\mathcal{P}_t = \begin{cases} \prod_{i=0}^{L-1} \frac{\pi_{t-i}}{\pi^*} & \text{if } \prod_{i=1}^{L} \pi_{t-i} < (\underline{\pi})^L, \\ \frac{\pi_t}{\pi^*} & \text{if } \prod_{i=1}^{L} \pi_{t-i} \geq (\underline{\pi})^L. \end{cases} \tag{26}$$

[32] The figure also includes a line indicating the boundary of the ZLB region.

[33] The asymmetric rule we study here responds to an accumulated shortfall of inflation (over a finite horizon), and therefore resembles features of the temporary price level targeting rules and threshold rules studied in, e.g., Bernanke, Kiley and Roberts (2019).

This policy targets average inflation (i.e. the product of gross inflation over the last L periods) when average inflation is less than $(\underline{\pi})^L$ and targets current inflation otherwise. Note that under the asymmetric rule the policymaker aims to overshoot the target, but does not aim to undershoot the target. More generally, the rule is a special case of a policy where the averaging window is state-dependent and extended when inflation is running persistently low.

If $\underline{\pi} < \pi^*$, then the asymmetric rule is a simple IT rule when average inflation is sufficiently near the level of the target. As a result, the asymmetric rule inherits the desirable local stability properties of the IT rule. On the other hand, the asymmetric rule is identical to the symmetric AIT rule near the target steady state if $\underline{\pi} > \pi^*$. Consequently, an asymmetric rule that targets average inflation when inflation is near the target poses risks to expectational stability.

Unsurprisingly, the asymmetric rule can fail to guide inflation back to target following a simulated liquidity trap if $\underline{\pi} > \pi^*$. In our calibrated full non-linear model, the asymmetric rule gives rise to persistent oscillations in inflation following a period of make-up inflation when $\underline{\pi} = 1.006 > \pi^* = 1.005$ (i.e. the annual inflation target is 2% but the central bank targets average inflation when inflation is persistently below about 2.4%).[34] Needless to say, an average inflation targeting central bank would abandon it's strategy before tolerating this undying pattern of over- and under-shooting the 2% target. The following remark summarizes these insights.

Remark 5 *Assume that the Taylor principle (8) holds for the asymmetric rule (25)-(26).*
(i) If $\underline{\pi} < \pi^*$, the target steady state of the model is locally robustly stable under learning with full opacity for all L and $\gamma \geq 0$.
(ii) If $\underline{\pi} > \pi^*$, the target steady state of the model is not robustly stable under learning with full opacity for any $L \geq 4$, and is locally unstable under full opacity if prices are fully flexible.

Remark 5 predicts that a fully opaque asymmetric rule will do a better job of bringing the economy out of the ZLB regime and back to the target steady state than a fully opaque symmetric rule – provided that the central bank focuses on guiding *current* inflation back to target when inflation is near or above the target.

For technical reasons, we exclude the case $\underline{\pi} = \pi^*$ in Remark 5, as the setup involves a dynamical system with regime switching and the fixed point on the boundary of two regimes. Various possibilities can arise in simulations and here are pertinent observations. $\underline{\pi} = \pi^*$ ensures that the policymaker does not aim to undershoot the target when average inflation is at or above the target level, and hence on the basis of simulations we conjecture that the results under $\underline{\pi} = \pi^*$ resemble those under $\underline{\pi} < \pi^*$. Figure IV illustrates the result from Remark 5 for the case $\underline{\pi} < \pi^*$ in the full non-linear model, assuming agents update beliefs using the non-linear version of (19). As before, we assume the economy is initially near the deflation steady state (i.e. $\pi_0^e \approx \pi_0 = \pi_{-1} = ... = \pi_{-L+1} \approx \pi_{Low}$, $y_0^e \approx y_0 \approx y_{Low}$, $R_0^e \approx R_0 \approx 1$), and we set $L = 6$, $\omega = 0.015$, and $\underline{\pi} = 1.001$.

Figure IV: ABOUT HERE

Figure IV displays results in terms of inflation for the asymmetric rule under full opacity (black line). For comparison the results under a symmetric IT rule (red line) and the

[34] Details are available on request.

benchmark symmetric AIT rule (4) under full opacity (orange line) are also shown in the figure. It is assumed that under both the asymmetric and symmetric AIT rules, makeup inflation with overshooting is observed after an initial period of low inflation, but dynamics under the rules differ when the downward adjustment starts. Under asymmetric AIT, overshooting of makeup inflation is very moderate and inflation does not undershoot the inflation target and, together with the interest rate, inflation gradually falls back to the target steady state. With the symmetric rule average inflation strongly overshoots the target and the policymaker abruptly raises the interest rate which makes both inflation and the interest rate undershoot the target and the dynamics become unstable.[35]

These results suggest that full opacity is no longer a concern if the policymaker uses an asymmetric rule of the form (25)-(26) that aims for overshooting, but not for undershooting of the target. Thus, while Hoffman et al. (2022) find that German households understand the implications of an asymmetric AIT strategy, our results suggest that asymmetric strategies can still perform well if expectations are not responsive to announcements about AIT, as Coibion et al. (2023) find in the case of U.S. households. Additional research on the performance of asymmetric makeup policy rules under adaptive learning would be worth while.

7 Variations on a Theme: Weighted Averages

We now consider whether the use of weighted measures of average inflation that discount past inflation relative to current inflation in computing average inflation can improve stability properties of AIT policy. We consider two natural deviations from the finite, simple moving averaging schemes studied above.

7.1 Exponentially Declining Weights

First, we introduce exponentially declining weights over the finite past horizon when computing average inflation for the interest rate rule. Thus the rule (4) is adjusted to

$$R_t \equiv 1 + \max[\bar{R} - 1 + \psi_p \left[\sum_{i=0}^{L-1} \mu^i (\frac{\pi_{t-i}}{\pi^*} - 1) \right] + \psi_y [\frac{y_t}{y^*} - 1], 0], \tag{27}$$

where $0 < \mu < 1$. The length of the past horizon is $L - 1$ as before. The framework is otherwise unchanged: the aggregate demand function (16), the Phillips curve (15) and learning with full opacity. The economy is stable under learning with full opacity and the rule (27) even when there is full price flexibility.

Proposition 7 *Assume that the Taylor principle (8) holds for the policy rule (27) for $0 < \mu < 1$. For small ω, the target steady state is locally stable under constant gain learning with full opacity for all L.*

Robustness of stability in Proposition 7 (ii) is examined using the calibrated model discussed in Section 3. Table IV repeats the analysis in Table I for different values of the discount parameter μ and under the assumption that $L = 6$.

[35]As noted above, the setup is sensitive to specific calibration details. For example, setting ω to a sufficiently small value can imply convergence to the target steady state under the symmetric rule with full opacity.

γ	42	128.21	350
ω_0 ($\mu = 1$)	0.00281	0.00417	0.00513
ω_0 ($\mu = .9$)	0.00569	0.00571	0.00615
ω_0 ($\mu = .7$)	0.01214	0.00936	0.00902
ω_0 ($\mu = .5$)	0.02059	0.01533	0.01424

Table IV: Least upper bounds ω_0 for instability

It is seen that discounting old data in the AIT rule contributes robustness of stability but a significant degree of discounting is needed. We conclude that this specification only modestly improves stability outcomes.

7.2 Exponential Moving Average Rule

A different way to discount old data is to assume that an exponential moving average specification is used in the interest rate rule. Consider an interest rate rule that responds to an exponential moving average of inflation (exp MA):

$$R_t = 1 + \max[\bar{R} - 1 + \psi_p \left(\frac{\pi_t^{w_c} (\pi_t^{cb})^{1-w_c}}{\pi^*} - 1 \right), 0] \quad (28)$$

$$\pi_t^{cb} = \pi_{t-1}^{w_c} (\pi_{t-1}^{cb})^{1-w_c}, \quad (29)$$

where $0 < w_c < 1$.[36] Intuitively, w_c determines an implicit averaging window, with higher values of w_c corresponding to shorter windows. The framework is otherwise unchanged: the aggregate demand function (16), the Phillips curve (15) and steady state learning.

The dynamic model is now given by

$$F(X_t, X_t^e, \pi_t^{cb}) = 0, \quad (30)$$

where F consists of the Phillips curve, the aggregate demand function and interest rate rule (28). The vector of current state variables is $X_t = (y_t, \pi_t, R_t)^T$. The law of motion for X_t^e is the same as before, and the law of motion for π_t^{cb} is given by (29). Linearizing around the target steady state we obtain the system

$$\hat{X}_t = (-DF_x)^{-1}(DF_{x^e}\hat{X}_t^e + DF_{cb}\hat{\pi}_t^{cb}) \quad (31)$$

$$\equiv M\hat{X}_t^e + N_{cb}\hat{\pi}_t^{cb}, \quad (32)$$

where M and N_{cb} are given in the appendix, and $\hat{X}$ again collects linearized y, π, R. In a model with sticky prices and an exponential moving average rule, the Taylor Principle is now sufficient for stability under constant gain learning with full opacity:

Proposition 8 *Assume that the Taylor principle (8) holds for the exponential moving average rule (28)-(29) and* $0 < w_c < 1$.
(i) Assume that there is price stickiness ($\infty > \kappa > 0$)*. For small* ω*, the target steady state is locally stable under constant gain learning with full opacity.*
(ii) Assume full price flexibility ($\kappa \to \infty$) *and* $\psi_p > \max[\frac{\pi^*(\frac{\omega}{w_c})(1-w_c)}{(1-\beta)\beta}, \beta^{-1}\pi^*]$*. For small* ω*, the target steady state is locally stable under constant gain learning with full opacity.*

[36] Budianto et al. (2023) study AIT with exp MA in a model with bounded rationality. Earlier, determinacy of REE under an exponential moving average rule was studied by Woodford (2003) (see Proposition 2.7).

When there is full price flexibility, however, the stability conditions depend on w_c and the ratio of ω to w_c, which implies that the situation can ultimately be more stringent than the preceding proposition indicates. If ω and w_c are both relatively small (i.e. the averaging window is relatively long) and $\omega \approx w_c$, then the condition for stability is far more demanding than the Taylor Principle. Eusepi and Preston (2018), section 4, study a related model that can be recovered here by setting $\omega = w_c$, and they obtain similar results. The fact that stability may depend on the private sector gain parameter suggests that the opaque exponential moving average formulation of average inflation targeting can be a risky alternative.

In Appendix C.5, it is shown that the REE under the exp MA rule is E-stable if agents embrace a PLM that includes a single lag of the interest rate. In these cases, E-stability means that agents can learn the averaging window, w_c. Furthermore, the target equilibrium is robustly stable under learning when agents have a PLM that includes a lag of the interest rate.

8 Concluding Remarks

Recent monetary policy challenges sparked interest in alternative policy frameworks, including AIT which the Federal Reserve adopted in 2020. The Fed has not communicated some details about the structure of the new policy, most notably the definition of average inflation itself. This paper explored some implications of imperfect knowledge in an AIT regime with significant history dependence.

We have focused on the stability criterion, i.e., convergence of the economy, including expectations, to the central bank's targeted equilibrium in the new policy regime. Our results indicate that implementation details of AIT should be carefully considered as there can be concerns about stability of the economy. AIT policy practiced under opacity of its details can fail to anchor expectations around the target steady state if prices are flexible or the speed of learning is anything but very slow. Moreover, an AIT policy practiced under opacity will typically fail to instigate an escape from a liquidity trap.

Silence about the details of its AIT regime, including the definition of average inflation can be problematic. Our research suggests that being opaque can result in instability of the economy. The instability result manifests the difficulty of stabilizing a simple finite moving of average inflation following a shock–which requires over- and under-shooting the long-run inflation target level–and anchoring expectations of agents who cannot correctly forecast these complicated implications of AIT. Robustness of the economy can be enhanced if certain limited aspects of the policy system are communicated to the private economy. In the model, AIT can anchor expectations if agents know the true averaging window or simply believe that the window is very long. The transparency about the policy regime required to induce these beliefs is only partial, as it is about the number of lags in the policy rule and knowledge of the numerical values of the rule parameters is not required for stability. If agents incorporate this information about the history-dependence of policy into their learning, then the target steady state is fairly robustly stable, and AIT can even succeed in guiding the economy out of a liquidity trap.

However, transparency is not always sufficient, as the adoption of an especially long averaging window precludes existence of a learning-stable equilibrium. In contrast, asymmetric strategies that aim for overshooting of the inflation target following a period of low inflation, but not undershooting of the target after a period of high inflation, can stabilize expectations under learning even with opacity about the policy rule and averaging window. Under such an asymmetric strategy, the length of the averaging window is not relevant for stability. An asymmetric approach prevents a difficult-to-forecast pattern of inflation overshooting followed by under-shooting the inflation target from emerging in equilibrium, and therefore helps anchor beliefs asymptotically.

There is plenty of room for future research. As a starting point for our analysis, we assumed that AIT is either conducted under full or partial opacity and, for comparison, in an environment in which agents fully incorporate knowledge of the structure of policy into learning. We highlight potential concerns about an opaque framework, but limited transparency can be beneficial in different ways that we did not consider here (e.g., see Faust and Svensson (2001), Geraats (2002), Jensen (2002), Jia and Wu (2023)). For instance, deviations from the rule may be called for in response to surprise structural shocks. In general, we neglected possible rationales for implementing an opaque AIT regime. However, this paper's findings also suggest benefits of adopting an opaque asymmetric rule. The performance of asymmetric rules, including switching rules, under imperfect knowledge is an area worth further exploring. Imperfect knowledge with learning may also have implications for optimal policy that have not yet been explored. Additionally, the analysis focused on the effects of an unanticipated implementation of AIT, and we have not studied anticipated transitions to AIT under conditions of imperfect knowledge.

Finally, our analysis about the success of AIT in stabilizing expectations assumed the presence of price adjustment costs à la Rotemberg (1982) in the economy. For brevity, we largely abstracted from alternative models of price stickiness, such as the widely used Calvo (1983) model of infrequent price-setting. The basic properties of the AIT in the Calvo model are given in Remark 3 and Appendix C.3. Further aspects of the comparison could be explored.

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Appendices

A The NK Model

The economic agents are household-producers, so the objective for agent s is to maximize expected, discounted isoelastic cum quadratic utility subject to a standard flow budget constraint (in real terms) over the infinite horizon. Households produce differentiated goods and s also denotes the product variety produced by s. The utility function for each period is standard except there is also disutility from changing prices:

$$U_{t,s} = \frac{c_{t,s}^{1-\sigma_1}}{1-\sigma_1} + \frac{\chi}{1-\sigma_2}\left(\frac{M_{t-1,s}}{P_t}\right)^{1-\sigma_2} - \frac{h_{t,s}^{1+\varepsilon}}{1+\varepsilon} - \frac{\gamma}{2}\left(\frac{P_{t,s}}{P_{t-1,s}} - 1\right)^2 \tag{33}$$

and the flow budget constraint is

$$st.\ c_{t,s} + m_{t,s} + \mathfrak{b}_{t,s} + \Upsilon_{t,s} = m_{t-1,s}\pi_t^{-1} + R_{t-1}\pi_t^{-1}\mathfrak{b}_{t-1,s} + \frac{P_{t,s}}{P_t}y_{t,s}. \tag{34}$$

Here $y_{t,s}$ and $c_{t,s}$ are production and consumption of agent s in period t. $M_{t-1,s}$ is the nominal balances in the beginning of the period and $h_{t,s}$ is labor supply of agent s in period t. $P_{t,s}$ is the price of product variety s while P_t is the aggregate price level. $m_{t,s}$, $\mathfrak{b}_{t,s}$ denote end of period real money balances and bond holdings of agent s. $\Upsilon_{t,s}$ is the lump-sum tax of agent s.

The final term in the utility function parameterizes the cost of adjusting prices in the spirit of Rotemberg (1982). The household decision problem is also subject to the usual "no Ponzi game" (NPG) condition. In the maximization of utility the expectations $E_{0,s}(.)$ are in general subjective and may not be rational.

Production function for good s is standard

$$y_{t,s} = h_{t,s}^{\alpha}, \text{ where } 0 < \alpha < 1. \tag{35}$$

There is no capital. Output is differentiated and firms operate under monopolistic competition. Each firm faces a downward-sloping demand curve

$$P_{t,s} = \left(\frac{y_{t,s}}{y_t}\right)^{-1/\nu} P_t. \tag{36}$$

Here $P_{t,s}$ is the profit maximizing price set by firm s consistent with its production $y_{t,s}$. The parameter ν is the elasticity of substitution between two goods and is assumed to be greater than one. y_t is aggregate output, which is exogenous to the firm. It is assumed that $\nu > 1$ and $\sigma = \sigma_1 \geq 1$.

The market clearing condition is

$$c_t + g_t = y_t.$$

The government consumes amount g_t of the aggregate good, collects the real lump-sum tax Υ_t from each consumer and issues bonds $\mathfrak{b}_t$ to cover financing needs. Fiscal policy is assumed to follow a linear tax rule for lump-sum taxes $\Upsilon_t = \kappa_0 + \kappa\mathfrak{b}_{t-1}$, where $\beta^{-1} - 1 < \kappa < 1$, so fiscal policy is "passive" using terminology of Leeper (1991). Government purchases

g_t are taken to be stochastic, so that $g_t = \bar{g} + \tilde{g}_t$, where the random part $\tilde{g}_t$ is an observable exogenous AR process

$$\tilde{g}_t = \rho \tilde{g}_{t-1} + \upsilon_t \tag{37}$$

with zero mean.[37]

A.1 Private sector optimization

Using the utility function of household-producer s (33) and the budget constraint (34) and production function (35), one computes the derivatives with respect to $(t-1)$-dated variables

$$\begin{aligned} \frac{\partial U_{t,s}}{\partial m_{t-1,s}} &= c_{t,s}^{-\sigma_1} \pi_t^{-1} + \chi (m_{t-1,s} \pi_t^{-1})^{-\sigma_2}, \\ \frac{\partial U_{t,s}}{\partial \mathfrak{b}_{t-1,s}} &= c_{t,s}^{-\sigma_1} R_{t-1} \pi_t^{-1}, \end{aligned}$$

and with respect to t-dated variables

$$\begin{aligned} \frac{\partial U_{t,s}}{\partial m_{t,s}} &= \frac{\partial U_{t,s}}{\partial \mathfrak{b}_{t,s}} = -c_{t,s}^{-\sigma_1}, \\ \frac{\partial U_{t,s}}{\partial P_{t,s}} &= c_{t,s}^{-\sigma_1} Y_t (1-\upsilon) \left(\frac{P_{t,s}}{P_t} \right)^{-\upsilon} \frac{1}{P_t} + \frac{\upsilon}{\alpha} h_{t,s}^{1+\varepsilon} \frac{1}{P_{t,s}}. \end{aligned}$$

The Euler equations are

$$\begin{aligned} \frac{\partial U_{t,s}}{\partial m_{t,s}} + \beta E_{t,s} \frac{\partial U_{t+1,s}}{\partial m_{t,s}} &= 0, \\ \frac{\partial U_{t,s}}{\partial \mathfrak{b}_{t,s}} + \beta E_{t,s} \frac{\partial U_{t+1,s}}{\partial \mathfrak{b}_{t,s}} &= 0, \\ \frac{\partial U_{t,s}}{\partial P_{t,s}} + \beta E_{t,s} \frac{\partial U_{t+1,s}}{\partial P_{t,s}} &= 0. \end{aligned}$$

The second equation is just the consumption Euler equation, while combining the first and second equations yields the money demand function. The third equation is the condition for optimal price setting .

Applying the above conditions, in period t each household s is assumed to maximize its anticipated utility $E_{0,s}(U_{t,s}(.))$ under given expectations. As in Evans et al. (2008), the first-order conditions for an optimum yield

$$\begin{aligned} 0 &= -h_{t,s}^{\varepsilon} + \frac{\alpha\gamma}{\nu} (\pi_{t,s} - 1) \pi_{t,s} \frac{1}{h_{t,s}} \\ &\quad + \alpha \left(1 - \frac{1}{\nu} \right) y_t^{1/\nu} \frac{y_{t,s}^{(1-1/\nu)}}{h_{t,s}} c_{t,s}^{-\sigma} - \frac{\alpha\gamma\beta}{\nu} \frac{1}{h_{t,s}} E_{t,s} (\pi_{t+1,s} - 1) \pi_{t+1,s}, \end{aligned} \tag{38}$$

$$c_{t,s}^{-\sigma} = \beta R_t E_{t,s} \left(\pi_{t+1}^{-1} c_{t+1,s}^{-\sigma} \right), \tag{39}$$

[37] For simplicity, it is assumed ρ is known (if not it could be estimated during learning). Only one shock is introduced to have a simple exposition.

where $\pi_{t+1,s} = P_{t+1,s}/P_{t,s}$ and $E_{t,s}(.)$ denotes the (not necessarily rational) expectations of agents s formed in period t.

Equation (38) is one form of the nonlinear New Keynesian Phillips curve describing the optimal price-setting by firms. The term $(\pi_{t,s} - 1)\,\pi_{t,s}$ arises from the quadratic form of the adjustment costs, and this expression is increasing in $\pi_{t,s}$ over the allowable range $\pi_{t,s} \geq 1/2$. Equation (39) is the standard Euler equation giving the intertemporal first-order condition for the consumption path.

We now write the decision rules for consumption and inflation so that they depend on forecasts of key variables over the infinite horizon (IH).

A.2 The Infinite-Horizon Phillips Curve

Starting with (38), let

$$Q_{t,s} = (\pi_{t,s} - 1)\,\pi_{t,s}. \tag{40}$$

The appropriate root for given Q is $\pi \geq \frac{1}{2}$ and so $Q \geq -\frac{1}{4}$ must be imposed to have a meaningful model. Using the production function $h_{t,s} = y_{t,s}^{1/\alpha}$ one can rewrite (38) as

$$Q_{t,s} = \frac{\nu}{\alpha\gamma} y_{t,s}^{(1+\varepsilon)/\alpha} - \frac{\nu - 1}{\gamma} y_t^{1/\nu} y_{t,s}^{(\nu-1)/\nu} c_{t,s}^{-\sigma} + \beta E_{t,s} Q_{t+1,s}, \tag{41}$$

and using the demand curve $y_{t,s}/y_t = (P_{t,s}/P_t)^{-\nu}$ gives

$$Q_{t,s} = \frac{\nu}{\alpha\gamma}(P_{t,s}/P_t)^{-(1+\varepsilon)\nu/\alpha} y_t^{(1+\varepsilon)/\alpha} - \frac{\nu-1}{\gamma} y_t (P_{t,s}/P_t)^{-(\nu-1)} c_{t,s}^{-\sigma} + \beta E_{t,s} Q_{t+1,s}.$$

Defining

$$x_{t,s} \equiv \frac{\nu}{\alpha\gamma}(P_{t,s}/P_t)^{-(1+\varepsilon)\nu/\alpha} y_t^{(1+\varepsilon)/\alpha} - \frac{\nu-1}{\gamma} y_t (P_{t,s}/P_t)^{-(\nu-1)} c_{t,s}^{-\sigma} \tag{42}$$

and iterating the Euler equation[38] yields

$$Q_{t,s} = x_{t,s} + \sum_{j=1}^{\infty} \beta^j E_{t,s} x_{t+j,s}, \tag{43}$$

provided that the transversality condition

$$\beta^j E_{t,s} x_{t+j,s} \to 0 \text{ as } j \to \infty \tag{44}$$

holds. It can be shown that (44) is an implication of the necessary transversality condition for optimal price setting. For further details see Benhabib et al. (2014).

The variable $x_{t+j,s}$ is a mixture of aggregate variables and the agent's own future decisions. Thus it provides only a "conditional decision rule". Equation (43) for $Q_{t,s}$ can be the basis for decision-making as follows.

So far only the agents' price-setting Euler equation and the above limiting condition (44) have been used. Some further assumptions are now made. Agents are assumed to have point expectations, so that their decisions depend only on the mean of their subjective forecasts. The model stipulates that all agents have the same utility and production functions. Initial money and debt holdings, and prices are assumed to be identical.

[38] It is assumed that expectations satisfy the law of iterated expectations.

The assumption of representative agents includes private agents' forecasting, so that the agents have homogenous forecasts of the relevant variables. Thus all agents make the same decisions at each point in time. It is also assumed that from the past agents have learned the market clearing relation in temporary equilibrium, i.e. $c_{t,s} = y_t - g_t$ in per capita terms and thus agents impose in their forecasts that $c^e_{t+j} = y^e_{t,t+j} - g^e_{t,t+j}$, where $g^e_{t,t+j} = \bar{g} + \rho^j \tilde{g}_t$. In the case of constant fiscal policy this becomes $c^e_{t+j} = y^e_{t+j} - \bar{g}$.

The assumption of representative agents implies that $P_{t,s} = P_{t,s'} = P_t$ for all agents s and s' in temporary equilibrium for all periods including the current one, see p. 224 in Benhabib et al. (2014). In that paper it was additionally assumed that agents' expectations also satisfy $P^e_{t+j,s} = P^e_{t+j}$ for future periods $j = 1, 2, ...$ This assumption is not necessary and is adopted here purely as a simplification.[39]

These considerations yield the infinite-horizon Phillips curve

$$\begin{aligned} Q_t \quad = \quad & \tilde{K}(y_t, y^e_{t+1}, y^e_{t+2}...) \equiv \frac{\nu}{\alpha\gamma} y_t^{(1+\varepsilon)\alpha^{-1}} - \frac{\nu-1}{\gamma}\frac{y_t}{(y_t - (\bar{g}+\tilde{g}_t))^\sigma} + \\ & \frac{\nu}{\gamma}\sum_{j=1}^{\infty} \alpha^{-1}\beta^j \left(y^e_{t+j}\right)^{(1+\varepsilon)\alpha^{-1}} - \frac{\nu-1}{\gamma}\sum_{j=1}^{\infty}\beta^j \frac{y^e_{t+j}}{(y^e_{t+j} - (\bar{g}+\rho^j\tilde{g}_t))^\sigma}. \end{aligned} \tag{45}$$

Its linearization is (6) in Section 2.2. Under steady state learning (45) becomes

$$\begin{aligned} \pi_t(\pi_t - 1) \quad = \quad & \tilde{K}(y_t, y^e_t) \equiv \frac{\nu}{\alpha\gamma} y_t^{(1+\varepsilon)\alpha^{-1}} - \frac{\nu-1}{\gamma}\frac{y_t}{(y_t-\bar{g})^\sigma} + \\ & \frac{\nu}{\gamma}\sum_{j=1}^{\infty}\alpha^{-1}\beta^j \left(y^e_t\right)^{(1+\varepsilon)\alpha^{-1}} - \frac{\nu-1}{\gamma}\sum_{j=1}^{\infty}\beta^j\frac{y^e_t}{(y^e_t-\bar{g})^\sigma} \text{ or} \\ \pi_t \quad = \quad & \Pi(y_t, y^e_t) \equiv Q_{-1}[\tilde{K}(y_t, y^e_t)]. \end{aligned}$$

A.3 The Consumption Function

To derive the consumption function from (39), use the flow budget constraint and the NPG condition to obtain an intertemporal budget constraint. Cashless limit is now assumed. First, define the asset wealth

$$\mathfrak{a}_t = \mathfrak{b}_t$$

as the holdings of real bonds and write the flow budget constraint as

$$\mathfrak{a}_t + c_t = y_t - \Upsilon_t + r_t\mathfrak{a}_{t-1}, \tag{46}$$

where $r_t = R_{t-1}/\pi_t$. Note that $(P_{jt}/P_t)y_{jt} = y_t$ is assumed, i.e. the representative agent assumption is invoked. Iterating (46) forward and imposing

$$\lim_{j\to\infty}(D^e_{t,t+j})^{-1}\mathfrak{a}^e_{t+j} = 0, \tag{47}$$

where

$$D^e_{t,t+j} = \frac{R_t}{\pi^e_{t+1}}\prod_{i=2}^{j}\frac{R^e_{t+i-1}}{\pi^e_{t+i}} \tag{48}$$

[39]More extensive discussion of the generalization is available in Evans, Honkapohja and Mitra (2022).

with $r^e_{t+i} = R^e_{t+i-1}/\pi^e_{t+i}$, one obtains the life-time budget constraint of the household

$$0 = r_t \mathfrak{a}_{t-1} + \Phi_t + \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} \Phi^e_{t+j} \quad (49)$$

$$= r_t \mathfrak{a}_{t-1} + \phi_t - c_t + \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} (\phi^e_{t+j} - c^e_{t+j}), \quad (50)$$

where

$$\begin{aligned} \Phi^e_{t+j} &= y^e_{t+j} - \Upsilon^e_{t+j} - c^e_{t+j}, \\ \phi^e_{t+j} &= \Phi^e_{t+j} + c^e_{t+j} = y^e_{t+j} - \Upsilon^e_{t+j}. \end{aligned} \quad (51)$$

Here all expectations are formed in period t, which is indicated in the notation for $D^e_{t,t+j}$ but is omitted from the other expectational variables.

Invoking the relations

$$c^e_{t+j} = (\beta^j D^e_{t,t+j})^{1/\sigma} c_t, \quad (52)$$

which are an implication of the consumption Euler equation (39), yields

$$c_t = r_t \mathfrak{a}_{t-1} + y_t - \Upsilon_t + \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} \phi^e_{t+j} - \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} (\beta^j D^e_{t,t+j})^{1/\sigma} c_t. \quad (53)$$

As we have $\phi^e_{t+j} = y^e_{t+j} - \Upsilon^e_{t+j}$, it follows that

$$c_t = \left(1 + \sum_{j=1}^{\infty} \beta^{j/\sigma} (D^e_{t,t+j})^{(1-\sigma)/\sigma}\right)^{-1} \left(r_t \mathfrak{b}_{t-1} + \sum_{j=0}^{\infty} (D^e_{t,t+j})^{-1} \phi^e_{t+j}\right).$$

So far it is not assumed that households act in a Ricardian way, i.e. they have not imposed the intertemporal budget constraint (IBC) of the government. To simplify the analysis, it is now assumed that consumers are Ricardian, which allows to modify the consumption function as in Evans and Honkapohja (2010). See Evans, Honkapohja and Mitra (2012) for discussion of the assumptions under which Ricardian Equivalence holds along a path of temporary equilibria with learning if agents have an infinite decision horizon.

The government flow constraint is

$$\mathfrak{b}_t + \Upsilon_t = \bar{g} + \tilde{g}_t + r_t \mathfrak{b}_{t-1} \text{ or } \mathfrak{b}_t = \Delta_t + r_t \mathfrak{b}_{t-1} \text{ where } \Delta_t = \bar{g} + \tilde{g}_t - \Upsilon_t.$$

By forward substitution, and assuming

$$\lim_{T \to \infty} (D^e_{t,t+T})^{-1} \mathfrak{b}^e_{t+T} = 0, \quad (54)$$

one gets

$$0 = r_t \mathfrak{b}_{t-1} + \Delta_t + \sum_{j=1}^{\infty} D^{-1}_{t,t+j} \Delta_{t+j}. \quad (55)$$

Note that Δ_{t+j} is the primary government deficit in $t+j$, measured as government purchases less lump-sum taxes. Under the Ricardian assumption, agents at each time t expect this constraint to be satisfied, i.e.

$$\begin{aligned} 0 &= r_t \mathfrak{b}_{t-1} + \Delta_t + \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} \Delta^e_{t+j}, \text{ where} \\ \Delta^e_{t+j} &= \bar{g} + \rho^j \tilde{g}_t - \Upsilon^e_{t+j} \text{ for } j = 1, 2, 3, \ldots. \end{aligned}$$

A Ricardian consumer assumes that (54) holds. His flow budget constraint (46) can then be written as:

$$\mathfrak{b}_t = r_t \mathfrak{b}_{t-1} + \psi_t, \text{ where } \psi_t = y_t - \Upsilon_t - c_t.$$

The relevant transversality condition is now (54). Iterating forward and using (52) together with (54) yields the consumption function and the aggregate demand function takes the form

$$y_t = g_t + c_t = g_t + \left(\sum_{j=1}^{\infty} \beta^{j/\sigma} (D^e_{t,t+j})^{(1-\sigma)/\sigma} \right)^{-1} \sum_{j=1}^{\infty} (D^e_{t,t+j})^{-1} (y^e_{t+j} - (\bar{g} + \rho^j \tilde{g}_t)), \qquad (56)$$

where the discount factor is given by (48).

A.4 Linearized IH Behavioral Rules

Linearizing (40) and (43) around the intended steady state and rearranging gives the following linearized expression for the Phillips curve:

$$\hat{\pi}_t = \kappa \hat{y}_t + \kappa \sum_{j=1}^{\infty} \beta^j \hat{y}^e_{t+j},$$

where $\hat{x}$ denotes a linearized variable, and κ is a complicated function of deep structural parameters.

The consumption function in (56) is linearized as follows. For the sake of brevity, assume $\tilde{g}_t = 0$. The discount factor $D^e_{t,t+j}$ has the linearization

$$\hat{D}^e_{t,t+j} = \beta^{1-j} \sum_{i=1}^{j} \left(\hat{R}^e_{t+i-1}/\pi^* - \hat{\pi}^e_{t+i}/(\beta\pi^*) \right).$$

After subtracting g_t from both sides of (56) and multiplying both sides of the resulting expression by $\sum_{j=1}^{\infty} \beta^{j/\sigma} (D^e_{t,t+j})^{(1-\sigma)/\sigma}$, linearizing the left-hand-side of the resulting consumption function gives

$$\begin{aligned} & \frac{\beta}{1-\beta}\hat{c}_t + c^* \frac{1-\sigma}{\sigma} \sum_{j\geq 1} \beta^{j/\sigma} \left(\beta^{-j}\right)^{(1-\sigma)/\sigma - 1} \hat{D}^e_{t,t+j} \\ = & \frac{\beta}{1-\beta}\hat{c}_t + c^* \frac{1-\sigma}{\sigma} \sum_{j\geq 1} \beta^{2j} \hat{D}^e_{t,t+j} \\ = & \frac{\beta}{1-\beta}\hat{c}_t + c^* \frac{1-\sigma}{\sigma} \sum_{j\geq 1} \beta^{j+1} \sum_{i=1}^{j} \left(\hat{R}^e_{t+i-1}/\pi^* - \hat{\pi}^e_{t+i}/(\beta\pi^*) \right). \end{aligned}$$

Linearizing the right-hand-side gives

$$\begin{aligned} & \sum_{j\geq 1} \beta^j \hat{y}^e_{t+j} - c^* \sum_{j\geq 1} \beta^{2j} \hat{D}^e_{t,t+j} \\ = & \sum_{j\geq 1} \beta^j \hat{y}^e_{t+j} - c^* \sum_{j\geq 1} \beta^{j+1} \sum_{i=1}^{j} \left(\hat{R}^e_{t+i-1}/\pi^* - \hat{\pi}^e_{t+i}/(\beta\pi^*) \right). \end{aligned}$$

Equating the two sides and rearranging gives linearized aggregate demand function

$$\hat{y}_t = -\frac{c^*\beta}{\sigma\pi^*}\hat{R}_t + \sum_{j=1}^{\infty}\beta^j\left(\frac{1-\beta}{\beta}\hat{y}^e_{t+j} - \frac{c^*}{\sigma}\left(\beta\hat{R}^e_{t+j}/\pi^* - \hat{\pi}^e_{t+j}/(\beta\pi^*)\right)\right). \tag{57}$$

The third equation is the linearized AIT interest rate rule (5).

A.5 Formulation of Learning

The basic model apart from the AIT rule is purely forward-looking while the observable exogenous shock $\tilde{g}_t$ is an $AR(1)$ process. Assuming full opacity about AIT rule, the appropriate PLM is a linear projection of $(y_{t+1}, \pi_{t+1}, R_{t+1})$ onto an intercept and the exogenous shock and agents estimate the regressions

$$s_u = a_s + b_s\tilde{g}_{u-1} + \varepsilon_{s,u}, \tag{58}$$

where $s = y$, π, R, by using a version of least squares and data for periods $u = 1, ..., t-1$. The latter is a common timing assumption in the learning literature; at the end of period $t-1$ the parameters of (58) are estimated using data through to period $t-1$. Usually, the estimation is done using recursive least squares. This gives estimates $a_{y,t-1}$, $b_{y,t-1}$, $a_{\pi,t-1}$, $b_{\pi,t-1}$, $a_{R,t-1}$, $b_{R,t-1}$ and using these estimates and data at time t the forecasts are given by

$$s^e_{t+j} = a_{s,t-1} + b_{s,t-1}\rho^j\tilde{g}_t,$$

for future periods $t+j$. These forecasts are then substituted into the system to determine a temporary equilibrium (also called the actual law of motion (ALM) of the economy in period t). With the new data point the estimates are updated and the process continues.

Denoting the PLM parameters by $\theta_t = (a_{y,t-1}, b_{y,t-1}, a_{\pi,t-1}, b_{\pi,t-1}, a_{R,t-1}, b_{R,t-1})$, the parameters are mapped into new values, so there is a mapping $\theta_t \to T(\ \theta_t)$ to the ALM parameters. The system consisting of temporary equilibrium and estimation equations is formally a *stochastic recursive algorithm* (SRA) and its convergence to equilibrium depends on the properties of $T(\theta)$. It should be noted that the SRA may be written in terms of decreasing or constant gain. The sense of probabilistic convergence is different in these two setups. Numerical analysis of this setup is done by simulating the SRA. It is possible to obtain analytical conditions for the stochastic convergence of the SRA to a fixed point θ^*. It turns out that in many cases conditions for convergence can be studied by examining the map $\theta \to T(\theta)$ and the ordinary differential equation

$$d\theta/d\tau = T(\theta) \tag{59}$$

in virtual time τ. Local stability conditions of a fixed point θ^* under (59) called **E-stability**, yield convergence conditions for the real-time SRA. Constant and decreasing gain learning are closely related. For example, in model (14) E-stability is established if stability under constant gain ω holds for all ω sufficiently small. For example, see Evans and Honkapohja (2001) or Evans and Honkapohja (2009a) for the theory and many applications.

It turns out that the technical analysis of convergence and computation of domains of attraction can be carried out using a simplification. Apart from the unknown policy rule the model is purely forward-looking while $\tilde{g}_t$ is an $AR(1)$ process. Under full opacity the PLM is a linear projection of the state variables $(y_{t+1}, \pi_{t+1}, R_{t+1})$ onto an intercept and the exogenous shock and in this case convergence of learning to a fixed point is fully governed by the dynamics of intercepts.

Thus, stability of a steady state can be validly assessed using the simplifying assumption that $\tilde{g}_t$ is identically zero. The agents are thought to estimate the long-run mean values of state variables, called "steady state learning". The latter is used here as a technical tool.

In simulations of the stochastic model agents are assumed to do least squares learning.

A.6 E-Stability for Linear Multivariate IH Models

Recall the system in first vector form

$$\tilde{X}_t = \tilde{K} + \sum\nolimits_{i=1}^{\infty} \beta^i \tilde{M} \tilde{X}^e_{t,t+i} + \tilde{N} \tilde{X}_{t-1}. \tag{60}$$

Consider first the learning with transparency case with PLM given by (11). The mapping PLM→ALM is (12). Assuming that the eigenvalues of $\tilde{A}$ and $\beta\tilde{A}$ are inside the unit circle, the mapping PLM→ALM simplifies to

$$\begin{aligned} \tilde{A} &\rightarrow \beta \tilde{M} \tilde{A}^2 \left(I - \beta \tilde{A}\right)^{-1} + \tilde{N} && (61) \\ \tilde{A}_0 &\rightarrow \tilde{K} + \beta \tilde{M} \left(I - \tilde{A}\right)^{-1} \left((1-\beta)^{-1} I - \tilde{A}^2 \left(I - \beta\tilde{A}\right)^{-1}\right) \tilde{A}_0. && (62) \end{aligned}$$

In this case it is straight-forward to obtain the E-stability conditions.

E-stability Conditions: Let $(\tilde{A}, \tilde{A}_0) = (\bar{A}, \bar{A}_0)$ denote a rational expectations equilibrium. The REE, $(\bar{A}, \bar{A}_0)$, is E-stable if the real parts of the eigenvalues of

$$\begin{aligned} DT(\tilde{A}) &= \left(\left(I - \beta\bar{A}\right)^{-1} \beta \bar{A}^2\right)^T \otimes \left(\tilde{M}\left(I - \beta\bar{A}\right)^{-1}\beta\right) + \\ &\quad I \otimes \left(\tilde{M}\left(I - \beta\bar{A}\right)^{-1}\beta\bar{A}\right) + \bar{A}^T \otimes \left(\tilde{M}\left(I - \beta\bar{A}\right)^{-1}\beta\right) && (63) \\ DT(\tilde{A}_0) &= \beta\tilde{M}\left(I - \tilde{A}\right)^{-1}\left((1-\beta)^{-1} I - \tilde{A}^2\left(I - \beta\tilde{A}\right)^{-1}\right) \end{aligned}$$

are less than one.

A.7 Model with Flexible Prices

In the special case of the NK model with flexible prices there is no Phillips curve and the first order condition (38) is replaced by the static condition

$$\frac{\partial U_{t,s}}{\partial P_{t,s}} = c_{t,s}^{-\sigma_1} y_t (1-\nu) \left(\frac{P_{t,s}}{P_t}\right)^{-\nu} \frac{1}{P_t} + \frac{\nu}{\alpha} h_{t,s}^{1+\varepsilon} \frac{1}{P_{t,s}} = 0.$$

Under symmetry it yields

$$c_t^{-\sigma_1} \alpha \frac{1-\nu}{\nu} + h_t^{1+\varepsilon-\alpha} = 0, \tag{64}$$

Steady-state learning with point expectations is formalized as before in Section 3. The temporary equilibrium equations with steady state learning are as follows.

1. With Ricardian consumers and assuming $\sigma = 1$ and $\tilde{g}_t = 0$, the market clearing equation is $y_t = g_t + c_t$ and yields

$$y_t = \bar{g} + (1-\beta)\left[y_t - \bar{g} + (y_t^e - \bar{g})\left(\frac{\pi_t^e}{R_t}\right)\left(\frac{R_t^e}{R_t^e - \pi_t^e}\right)\right] \tag{65}$$

as the aggregate demand relation.

2. The static labor-consumption optimality condition (64) can be combined with market clearing to obtain

$$y_t = \left(\alpha\frac{\nu-1}{\nu}(y_t - g_t)^{-\sigma_1}\right)^{\alpha/(1+\varepsilon-\alpha)}. \tag{66}$$

Looking at (66) it is evident that output in temporary equilibrium is exogenous.[40]

3. Interest rate rule (1) as discussed in the text.

If one substitutes the interest rate rule (1) and also an exogenous value of output into (65), the model effectively says that the nominal interest rate R_t (and π_t via the policy rule) is the variable that establishes equality of aggregate demand and supply in temporary equilibrium. Using the interest rate rule (1) then yields the temporary equilibrium value for inflation π_t.

B The ALM When There is Underparameterization

B.1 General setup

The model (10) is now modified to include an underparameterized PLM

$$\begin{aligned}
\tilde{X}_{1,t} &= A_0 + \tilde{A}_N \tilde{X}_{1,t-1}, \text{ where } N < L, \qquad (67)\\
\tilde{X}_{1,t} &= \begin{pmatrix} \hat{X}_t \\ \vdots \\ \hat{X}_{t-(N-2)} \end{pmatrix} \text{ and } \tilde{X}_t = \begin{pmatrix} \tilde{X}_{1,t} \\ \tilde{X}_{2,t} \end{pmatrix},\\
\tilde{A}_N &= \begin{pmatrix} A_1 & \cdots & A_{N-2} & A_{N-1} \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix} \text{ and } \tilde{A}_0 = \begin{pmatrix} A_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\end{aligned}$$

The vector $\tilde{X}_{1,t}$ is $3(N-1)$ dimensional and it consists of the current and first $N-2$ lagged endogenous variables. Thus the first block of $\tilde{X}_{1,t}$ is

$$\hat{X}_t = A_0 + \sum_{i=1}^{N-1} A_i \hat{X}_{t-i}, \tag{68}$$

while the other blocks express identities. Here $\tilde{A}_N$ is a square matrix with dimensions $3(N-1)$, where $N-1$ is the number of lags in the PLM of agents. Matrices A_i are 3×3, while $\tilde{A}_0$ and A_0 are $3(N-1)$ and 3 dimensional column vectors, respectively.

[40] Exogeneity of output holds in the classical monetary model, see e.g. Gali (2008), chapter 1.

Iterating the PLM, we get

$$\tilde{X}^e_{1,t,t+i} = (I + \tilde{A}_N + ... + \tilde{A}^i_N)\tilde{A}_0 + \tilde{A}^{i+1}_N \tilde{X}_{1,t-1}$$

and the unprojected ALM is

$$\begin{pmatrix} \tilde{X}_{1,t} \\ \tilde{X}_{2,t} \end{pmatrix} = \left(\begin{bmatrix} \tilde{K} + \bar{M}_{11} \sum_{i=1}^{\infty} \beta^i [(I + \tilde{A}_N + ... + \tilde{A}^i_N)\tilde{A}_0 + \tilde{A}^{i+1}_N \tilde{X}_{1,t-1}] \\ 0 \end{bmatrix} + \begin{pmatrix} \hat{N} & \hat{N} & \cdots & \hat{N} \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} \hat{X}_{t-1} \\ \hat{X}_{t-2} \\ \vdots \\ \hat{X}_{t-(L-1)} \end{pmatrix} \right),$$

where $\bar{M}_{11}$ is the submatrix of $\tilde{M}$ formed from the first $3(N-1)$ rows and columns of $\tilde{M}$ in (10). The matrix multiplying $(\hat{X}^T_{t-1}, ..., \hat{X}^T_{t-(L-1)})^T$ is as before, see proof of Proposition 1. It is seen that the ALM is not in the same space as the PLM as there are further lags of $\hat{X}_t$ in the ALM. The map of the intercept term is

$$\begin{aligned} A_0 &\rightarrow \tilde{K} + \bar{M}_{11} \sum_{i=1}^{\infty} \beta^i (I + \tilde{A}_N + ... + \tilde{A}^i_N)\tilde{A}_0 \\ &= \tilde{K} + \bar{M}_{11} \left(\sum_{i=1}^{\infty} \beta^i (I - \tilde{A}_N)^{-1}(I - \tilde{A}^{i+1}_N) \right)\tilde{A}_0 \\ &= \tilde{K} + \bar{M}_{11}(I - \tilde{A}_N)^{-1}[\frac{\beta}{1-\beta} I - \beta \tilde{A}^2_N (I - \beta \tilde{A}_N)^{-1}]\tilde{A}_0, \end{aligned} \tag{69}$$

if the eigenvalues of $\tilde{A}_N$ and $\beta\tilde{A}_N$ are inside the unit circle.

Noting that

$$\bar{M}_{11}\beta\tilde{A}^2_N \sum_{i=0}^{\infty} \beta^i \tilde{A}^i_N = [\bar{M}_{11}\beta\tilde{A}^2_N (I - \beta\tilde{A}_N)^{-1}],$$

the autoregressive term of the unprojected ALM can be written

$$\begin{pmatrix} \tilde{X}_{1,t} \\ \tilde{X}_{2,t} \end{pmatrix} = \begin{pmatrix} \bar{M}_{11}\beta\tilde{A}^2_N (I - \beta\tilde{A}_N)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{X}_{1,t-1} \\ \tilde{X}_{2,t-1} \end{pmatrix} + \begin{pmatrix} \hat{N}(\hat{X}_{t-1} + ..., + \hat{X}_{t-(L-1)}) \\ \hat{X}_{t-2} \\ \vdots \\ \hat{X}_{t-(L-1)} \end{pmatrix} + Z_t. \tag{70}$$

where Z_t is a white noise disturbance term. Define

$$F(\tilde{A}_N) = [\bar{M}_{11}\beta\tilde{A}^2_N (I - \beta\tilde{A}_N)^{-1}]$$

is $3(N-1) \times 3(N-1)$ while the dimension of the whole system is $3(L-1) \times 3(L-1)$. Note also the form of $\bar{M}_{11}$, which is zero except for top-left 3×3 corner. The (unprojected) ALM is $VAR(L-1)$ in the vector $\hat{X}_t$ while the PLM is a $VAR(N-1)$ process, and it is necessary to map the unprojected ALM into the space of $VAR(N-1)$ processes.

Next, compute

$$F(\tilde{A}_N) = \begin{pmatrix} \beta M(A_1^2 + A_2) & \cdots & \beta M(A_1 A_{N-2} + A_{N-1}) & \beta M A_1 A_{N-1} \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \tag{71}$$

$$\times \left(I - \beta \begin{pmatrix} A_1 & A_2 & \cdots & A_{N-2} & A_{N-1} \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix} \right)^{-1}$$

Looking at this form of $F(\tilde{A}_N)$, we note that in the first matrix all except the first row blocks of 3×3 matrices are zero. This allows writing $F(\tilde{A}_N)$ in the form

$$F(\tilde{A}_N) = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 & \cdots & \mathcal{M}_{N-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{72}$$

and the unprojected ALM is

$$\hat{X}_t^{ALM} = (\mathcal{M}_1 + \hat{N})\hat{X}_{t-1} + ... + (\mathcal{M}_{N-1} + \hat{N})\hat{X}_{t-(N-1)} + \hat{N}\hat{X}_{t-N} + ... + \hat{N}\hat{X}_{t-(L-1)} + Z_t. \tag{73}$$

Next, denote the projected ALM as

$$\hat{X}_t^{PR} = \sum_{i=1}^{N-1} B_i \hat{X}_{t-i} + \varepsilon_t.$$

Then the mean forecast error is $E[\hat{X}_t^{ALM} - \hat{X}_t^{PR}] = E[ES]$ where

$$ES = \sum_{i=1}^{N-1} (\mathcal{M}_i + \hat{N} - B_i)\hat{X}_{t-i} + \hat{N}(\hat{X}_{t-N} + ... + \hat{X}_{t-(L-1)}).$$

Minimizing the square of the mean forecast error yields the orthogonality conditions:

$$0 = E[(ES)\hat{X}_{t-j}^T] = \sum_{i=1}^{N-1} E[(\mathcal{M}_i + \hat{N} - B_i)\hat{X}_{t-i}\hat{X}_{t-j}^T + \hat{N}(\hat{X}_{t-N} + ... + \hat{X}_{t-(L-1)})\hat{X}_{t-j}^T],$$

for $j = 1, ...N-1$.

Now let $\hat{X}_{t-i} = \left(\hat{y}_{t-i}, \hat{R}_{t-i}, \hat{\pi}_{t-i}\right)^T$ and define[41]

$$E\hat{X}_{t-i}\hat{X}_{t-j}^T = \begin{pmatrix} E(\hat{y}_{t-i}\hat{y}_{t-j}) & E(\hat{y}_{t-i}\hat{R}_{t-j}) & E(\hat{y}_{t-i}\hat{\pi}_{t-j}) \\ E(\hat{R}_{t-i}\hat{y}_{t-j}) & E(\hat{R}_{t-i}\hat{R}_{t-j}) & E(\hat{R}_{t-i}\hat{\pi}_{t-j}) \\ E(\hat{\pi}_{t-i}\hat{y}_{t-j}) & E(\hat{\pi}_{t-i}\hat{R}_{t-j}) & E(\hat{\pi}_{t-i}\hat{\pi}_{t-j}) \end{pmatrix} = \Omega_{i.j}, \tag{74}$$

[41] This standard procedure presumes that $\hat{X}_{t-i}$ is covariance stationary. However, the algebraic operation can be done even if stationarity is not presupposed.

Then

$$0 = E[(ES)\hat{X}_{t-j}^T] = \sum_{i=1}^{N-1}(\mathcal{M}_i + \hat{N} - B_i)\Omega_{i,j} + \hat{N}\sum_{k=N}^{L-1}\Omega_{k,j}, \text{ for } j = 1, ...N-1. \tag{75}$$

We also need to use the Yule-Walker equations to compute $\Omega_{k,j}(m,n)$ for $k, j = 1, ..., L-1$ and $m, n = 1, 2, 3$.[42] Let Ω_{kj} denote the 3×3 matrix with elements $\Omega_{k,j}(m,n)$. Then use (72) and define

$$F_{ext}(\tilde{A}_N) = \begin{pmatrix} \mathcal{M}_1 + \hat{N} & \mathcal{M}_2 + \hat{N} & \cdots & \mathcal{M}_{N-1} + \hat{N} & \hat{N} & \cdots & \hat{N} & \hat{N} \\ I & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & I & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & I & 0 \end{pmatrix}$$

which is $3(L-1) \times 3(L-1)$ matrix. Consider its eigenvalues from the system

$$\det[F_{ext}(\tilde{A}_N) - \lambda I] = 0.$$

Then define

$$\mathfrak{A} = F_{ext}(\tilde{A}_N) \otimes F_{ext}(\tilde{A}_N)$$

and obtain the matrix of second moments of the $VAR(L-1)$ process (70). in vector form

$$vec(\Sigma) = (I - \mathfrak{A})^{-1} vec(Q_Z),$$

where $\Sigma = (\omega_{ij}) = E(\hat{X}_t^{ALM}(\hat{X}_t^{ALM})^T)$ and Q_Z is the covariance matrix of the augmented form of the error term Z_t. Alternatively, one can use the linear equation system

$$\Sigma = F_{ext}(\tilde{A}_N)\Sigma F_{ext}(\tilde{A}_N)^T + Q_Z.$$

Introducing the notation

$$\Sigma = \begin{pmatrix} \Gamma_0 & \Gamma_1 & \cdots & \Gamma_{L-2} \\ \Gamma_1^T & \Gamma_0 & \cdots & \Gamma_{L-3} \\ \vdots & \vdots & \cdots & \vdots \\ \Gamma_{L-2}^T & \Gamma_{L-3}^T & \cdots & \Gamma_0 \end{pmatrix},$$

we have $\Gamma_i = E(\hat{X}_t^{ALM}(\hat{X}_{t-i}^{ALM})^T)$ for $i = 1, ..., L-2$. The $i'th$ autocovariance matrix of the original process (70) is then

$$\Gamma_i = (\mathcal{M}_1 + \hat{N})\Gamma_{i-1} + ... + (\mathcal{M}_{N-1} + \hat{N})\Gamma_{i-(N-1)} + \hat{N}\Gamma_{i-N} + ... + \hat{N}\Gamma_{i-(L-1)}$$

for $i = L-1, L, ...$ from which the required covariances $\Omega_{kj}(m,n)$ for $k, j = 1, ..., L-1$ and $m, n = 1, 2$ are obtained and substituted into (74). As $\Omega_{ij} = E\hat{X}_{t-i}\hat{X}_{t-j}^T$ we have

$$\Gamma_{j-i} = \Omega_{ij} = \begin{cases} \Gamma_{j-i} \text{ for } j \geq i \\ \Gamma_{j-i}^T \text{ for } j < i. \end{cases}.$$

These equations are used in the numerical analysis reported in Table II and in Appendix B.3.

[42] The method is explained well in Section 10.2 of Hamilton (1994).

B.2 Stability in the sticky price case

We begin with the technical details for Remark 4 and Table II in the case of PLM lag length, $N-1$, where $1 < N \leq L-1$. For simplicity, we assume that exogenous shocks are mean-zero and *i.i.d*, and that agents have the following PLM

$$\begin{aligned}\tilde{X}_t &= \tilde{A}_N \tilde{X}_{t-1} + \tilde{A}_{0,t-1} \\ \tilde{A}_{0,t} &= \omega(\tilde{X}_t - \tilde{A}_N \tilde{X}_{t-1} - \tilde{A}_{0,t-1}) + \tilde{A}_{0,t-1}.\end{aligned}$$

$\tilde{A}_N$ contains the RPE coefficients (or the correct ALM coefficients in the case $N = L$), imposes zeros on lags of $\hat{X}$ exceeding $N-1$, and is fixed over time (i.e. agents only estimate the intercept term recursively). The actual law of motion for $\tilde{X}_t$ can be expressed as

$$\tilde{X}_t = B\tilde{X}_{t-1} + DT_a \tilde{A}_{0,t-1} + shocks$$

where B and DT_a are functions of $\tilde{A}_N$ and the model's structural parameters. This implies the following law of motion for $W_t = (\tilde{X}_t^T, \tilde{A}_{0,t}^T)^T$

$$\begin{aligned}W_t &= \begin{pmatrix} B & DT_a \\ \omega(B - \tilde{A}_N) & \omega DT_a + (1-\omega)I \end{pmatrix} W_{t-1} + shocks \\ &= DT * W_{t-1} + shocks\end{aligned}$$

We solve for largest gain parameter, ω_0, that ensures that the roots of DT are inside the unit circle. Robust stability obtains if $\omega_0 > 0.01$, i.e. we have stability under constant gain learning for all $\omega < 0.01$. The numerical results are given in Table II in the main text. For those calculations, we use the calibration reported in section 3 with $\gamma = 128.21$. For simplicity, the shock is assumed to be *i.i.d.*[43]

B.3 Flexible price case

Consider the linearized IH model of the flex-price economy studied above. Output is exogenous, so (7) and (5) yield the system

$$\begin{pmatrix} \hat{R}_t \\ \hat{\pi}_t \end{pmatrix} = \sum_{j=1}^{\infty} \beta^j \begin{pmatrix} -1 & \beta^{-2} \\ -\frac{\pi^*}{\psi_p} & \beta^{-2}\frac{\pi^*}{\psi_p} \end{pmatrix} \begin{pmatrix} \hat{R}^e_{t+j} \\ \hat{\pi}^e_{t+j} \end{pmatrix} + \sum_{i=1}^{L-1} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \hat{R}_{t-i} \\ \hat{\pi}_{t-i} \end{pmatrix} + \begin{pmatrix} \epsilon_t \\ \frac{\pi^*}{\psi_p}\epsilon_t \end{pmatrix}, \tag{76}$$

where an *i.i.d.* shock to the aggregate demand has been added. This model can be put in the form (73), where the coefficient matrices are 2×2 and take the form[44]

$$M = (m_{ij}),\ \hat{N} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \mathcal{A}_i \equiv \mathcal{M}_i + \hat{N} = \begin{pmatrix} 0 & a_{i12} \\ 0 & a_{i22} \end{pmatrix} \text{ for } i = 1, \ldots, N-1, \tag{77}$$

where $m_{11} = -1$, $m_{12} = \beta^{-2}$, $m_{21} = \pi^*/\psi_p$, $m_{22} = -\beta^{-2}\pi^*/\psi_p$. Postulating the PLM with matrices A_i, $i = 1, N-1$, it is possible in principle to compute the unprojected and projected ALMs. For example, we can show that both ALMs yield non-stationary processes for any set of PLM parameters when the model is calibrated with $\beta = 0.99$, $\psi_p = 1.2$, $\pi^* = 1.005$,

[43] Mathematica routine available on request.

[44] In the flex-price case the first column of $\mathcal{M}_i$ and $\hat{N}$ are zero.

$L = 4$ and $N = L - 1$. With these parameter values and augmenting the ALM with $\hat{N}$ to $VAR(3)$ process it is seen that the projected ALM process is non-stationary.

Proposition 3 in Section 3.4 establishes non-stationarity under more general assumptions about the flexible price model. The proposition covers the general case $N < L - 1$. The proof of the proposition is in Appendix D.

C Further Results

C.1 Domain of Escape for Inflation Targeting and AIT

Figure A.1 shows the domain of escape under IT. The basic parameter settings are as given earlier.

Figure A.1 HERE

It should be noted that the result about escape from low steady state π_{Low}, y_{Low} differs from that in Figure I of Honkapohja and Mitra (2020). There are some differences in parameter values and most importantly in initial conditions for R_0 and R_0^e. In computing conditional domain of attraction it is natural to assume that R_0 and R_0^e are approximately equal to the steady state value R^*, whereas computation of domain of escape Figure A.1 assumes that R_0 and R_0^e are approximately 1.

Figure A.2 shows the domain of escape under AIT when $\tilde{A}_0 = \bar{A}$ where $\bar{A}$ is the rational expectations equilibrium coefficients corresponding to the unique dynamically stable MSV solution of the linearized model (i.e. the model linearized around the target steady state).

Figure A.2 HERE

C.2 Robust Stability: Policy Parameters and Transparency

Tables C.1 and C.2 show how Table I results (full opacity) change if either $\psi_p = 2$ and $\psi_y = 0.125$, or $\psi_p = 1.5$ and $\psi_y = 1$. Table C.3 reports robust stability results for the case of learning with transparency. The approach outlined in Appendix B.2 with $N = L$ and $\tilde{A}_n$ containing the correct MSV coefficients was taken to produce the results reported in Table C.3.

γ	42	128.21	350
ω_0 (IT)	0.03104	0.03255	0.03642
ω_0 (PLT)	0.01164	0.00742	0.00432
ω_0 (AIT with $L = 6$)	0.00262	0.00391	0.00493
ω_0 (AIT with $L = 20$)	0.00020	0.00041	0.00075
ω_0 (AIT with $L = 32$)	0.00008	0.00016	0.00032

Table C.1: Least upper bounds for $\psi_p = 2$ with full opacity

γ	42	128.21	350
ω_0 (IT)	0.03058	0.03920	0.04337
ω_0 (PLT)	0.00991	0.00723	0.00505
ω_0 (AIT with $L = 6$)	0.00381	0.00552	0.00752
ω_0 (AIT with $L = 20$)	0.00035	0.00076	0.00134
ω_0 (AIT with $L = 32$)	0.00014	0.00032	0.00062

Table C.2: Least upper bounds for $\psi_y = 1$ with full opacity

γ	42	128.21	350
ω_0 ($L = 6$)	0.02159	0.02360	0.02662
ω_0 ($L = 20$)	0.02163	0.02389	0.02764
ω_0 ($L = 32$)	0.02163	E-unstable	E-unstable

Table C.3: Least upper bounds under learning with transparency.

Rule	*IT*	*AIT* ($L = 6$)	*AIT* ($L = 20$)	*AIT* ($L = 32$)
ω_0	0.29358	0.01669	0.00285	0.00154

Table C.4: Least upper bounds for $\kappa \approx 0.005$ with full opacity[45]

C.3 Robustness of Stability in the Calvo Model

Table C.5 gives the least upper bounds for robustness of stability in the Calvo model. Following Preston (2005) or more recently Eusepi, Gibbs and Preston (2021), the two behavioral equations, log-linearized at the target steady state, are[46]:
(i) the aggregate demand curve

$$\hat{y}_t = -\sigma \hat{\imath}_t + \hat{E}_t \sum_{i=0}^{\infty} \beta^i [(1-\beta)\hat{y}_{t+1+i} - \sigma(\beta \hat{\imath}_{t+1+i} - \hat{\pi}_{t+1+i})] + r_t^n,$$

where r_t^n is a demand shock.
(ii) the Phillips curve

$$\hat{\pi}_t = \kappa \hat{y}_t + \hat{E}_t \sum_{i=0}^{\infty} (\alpha\beta)^i [\kappa\alpha\beta \hat{y}_{t+1+i} + (1-\alpha)\beta \hat{\pi}_{t+i+1}].$$

Here hat denotes proportional deviation of the variable from its value at the target steady state. Further, β is the subjective discount rate, σ is the intertemporal elasticity of substitution, κ is a parameter indicating degree of price stickiness and α is the fraction of firms which cannot change the price in a period. We tacitly assume log-utility in consumption and linear disutility in labor, so that $\kappa = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}$. There is also
(iii) the linearized AIT interest rate rule[47]

$$\hat{\imath}_t = \bar{\psi}_p \left(\sum_{k=0}^{L-1} \hat{\pi}_{t-k} \right) + \psi_y \hat{y}_t.$$

It is straightforward to show that the analogue of Proposition 1 holds.[48] For robustness of stability (analogue of Remark 1) table C.5 reports robust stability results under learning with opacity for IT, AIT, PLT and for different values of α. All other parameters are reported in Section 2.3.

[45]In Table C.4, we set $\gamma = 17000$ but leave all other parameters at benchmark values.
[46]See Preston (2005), equations (18) and (19).
[47]Note that $\hat{\imath}_t = (i_t - i^*)/i^* = \hat{R}_t(1+i^*)/i^*$ as $R = 1+i$. So $\bar{\psi}_p = \psi_p(i^*/(1+i^*))$, where ψ_p is the policy rule parameter in the undeviated model.
[48]Details are available on request.

α	0.66	0.75	0.9
ω_0 (IT)	0.13618	0.15681	0.17895
ω_0 (PLT)	0.03467	0.20753	0.04287
ω_0 $(AIT$ with $L = 6)$	0.01158	0.01938	0.10152
ω_0 $(AIT$ with $L = 20)$	0.00166	0.00265	0.01044
ω_0 $(AIT$ with $L = 32)$	0.00073	0.00121	0.00439

Table C.5: Least upper bounds for $\psi_p = 1.5$, $\psi_y = 0.125$

We have thus shown that the full opacity results under Calvo and Rotemberg are similar for a standard calibration. Naturally, one could study further stability and other properties for the Calvo model in the same way as was done in this paper. We conjecture that analogues of various results shown above for the Rotemberg model also hold for the Calvo model.

C.4 Price level targeting vs. AIT

Consider the case of full opacity in the linearized economy described in section 3.2 and fix the gain to zero ($\omega = 0$). Under PLT, inflation (π) and the price gap (p) follow the processes:[49]

$$\begin{aligned} \hat{\pi}_t &= -\pi^* h \hat{p}_{t-1} \\ \hat{p}_t &= (1-h)\hat{p}_{t-1} = (\pi^*)^{-1}\hat{\pi}_t + \hat{p}_{t-1} \end{aligned}$$

where

$$h = \frac{\beta\kappa\psi_p y^*(y^* - \bar{g})}{\beta(\pi^*)\psi_y(y^* - \bar{g}) + \beta\kappa\psi_p y^*(y^* - \bar{g}) + (\pi^*)^2\sigma y^*} \in [0, 1].$$

These expressions for inflation and the price level gap under PLT with full opacity and zero gain reveal several properties of the policy regime:

1. Given $(1 - h) \in [0, 1]$, the price gap converges monotonically to the target provided that $\kappa > 0$. Hence, if PLT is implemented when prices are undesirably low ($\hat{p}_{-1} < 0$), then the price level converges monotonically to zero from below.

2. Because price gap converges monotonically to zero, so does inflation. E.g., if $\hat{p}_{-1} < 0$, then $\hat{\pi}_{t-1} \geq \hat{\pi}_t \geq 0$ for all $t \geq 1$ and we have $\lim_{t\to\infty} \hat{\pi}_t = 0$.

3. As prices become very sticky (κ close to zero), the "make-up" inflation implied by the policy regime becomes more persistent. In the special case of fully flexible prices, $\hat{\pi}_0$ adjusts to fully close the price level gap and then $\hat{\pi}_t = \hat{p}_t = 0$ for $t > 0$. In this regard, price flexibility enhances the performance of PLT.

Consider now AIT with full opacity and gain equal to zero. Inflation and the implied price level gap follow the processes:

$$\hat{\pi}_t = -h \sum_{k=1}^{L-1} \hat{\pi}_{t-k} = -\pi^* h(\hat{p}_{t-1} - \hat{p}_{t-L})$$

[49]The linearized PLT rule is: $\hat{R}_t = \psi_p \hat{p}_t + \frac{\psi_y}{y^*}\hat{y}_t$. For brevity, we set exogenous shocks to zero in each period and assume $\hat{\pi}^e_{-1} = \hat{R}^e_{-1} = \hat{y}^e_{-1} = 0$ throughout section C.4, but the main results of this section do not hinge of these details.

and

$$\hat{p}_t = (1-h)\hat{p}_{t-1} + h\hat{p}_{t-L} = (\pi^*)^{-1}\hat{\pi}_t + \hat{p}_{t-1}$$

where h is defined above. In first-order form:

$$X_t = AX_{t-1}$$

where $X_t = (\hat{\pi}_t, ..., \hat{\pi}_{t-L+2})^T$. The roots of A solve the characteristic polynomial: $\lambda^{L-1} + h\sum_{k=0}^{L-2}\lambda^k$. It is possible to show that the roots of A are strictly inside the unit circle if $\kappa < \infty$ (price are sticky), but there are the roots of unity if prices are fully flexible.[50] In the case of flexible prices specifically, we have that $h = 1$, which implies:

$$\hat{\pi}_t = -\sum_{k=1}^{L-1}\hat{\pi}_{t-k} = -\pi^*(\hat{p}_{t-1} - \hat{p}_{t-L})$$

and

$$\hat{p}_t = \hat{p}_{t-L}$$

which implies: $\hat{\pi}_t = \hat{\pi}_{t-L}$. Studying these equations reveal some important properties of AIT that separate the finite window AIT specification from PLT:

1. In the case of AIT, inflation and the implied price level gap follow deterministic cycles under fully flexible prices, and they tend to oscillate under sticky prices. This is in stark contrast to PLT, which ensures that inflation converges monotonically to the target to close an initial price level gap.

2. Increasing price flexibility worsens stability under AIT and leads to a situation where an initial inflation gap is never closed. In contrast, price flexibility implies that the price level gap is closed immediately under PLT.

Simulations illustrate key differences between PLT under opacity and AIT under opacity with $\omega = 0$, see Figures A.3 and A.4 below. The simulations assume that annual inflation is persistently 1% below target for roughly 8 years prior to the implementation of AIT or PLT. It is seen that inflation overshoots the target under PLT ("make-up" inflation) before monotonically converging to target. The convergence happens in only one period under PLT with flexible prices. Under AIT, however, there are oscillations. With flexible prices, these oscillations are deterministic cycles. Numerically, it can be shown that while $\omega = 0$, increasing L in the sticy price model causes the largest root determining the stability of the system to increase, which suggests that instability will result for smaller values of $\omega > 0$ when L is large, as confirmed by Tables 1, C.1-C.2, C.4-C.5.

FIGURES A.3 AND A.4 ABOUT HERE

C.5 Learning the Exponential Moving Average Rule

In section 4 we considered the issue of whether agents can learn the window length L under a finite simple moving average AIT regime. It turns out that the agents can learn w_c in an opaque exponential moving average AIT regime as well, but only if they use a specific

[50] The result is shown using the Jury stability criterion. See the proof of Proposition 1.

form of PLM. To see this, consider the following approximation of the exp MA rule (28) and (29):[51]

$$\hat{R}_t = \frac{\psi_p}{\pi^*}\left(w_c\hat{\pi}_t + w_c\frac{\psi_y\pi^*}{\psi_p y^*}\hat{y}_t + (1-w_c)\hat{\pi}_t^{cb}\right) \tag{78}$$
$$\hat{\pi}_t^{cb} = w_c\hat{\pi}_{t-1} + w_c\frac{\psi_y\pi^*}{\psi_p y^*}\hat{y}_{t-1} + (1-w_c)\hat{\pi}_{t-1}^{cb}.$$

Equivalently:[52]

$$\hat{R}_t = w_c\left(\frac{\psi_p}{\pi^*}\hat{\pi}_t + \frac{\psi_y}{y^*}\hat{y}_t\right) + (1-w_c)\hat{R}_{t-1}. \tag{79}$$

It can be shown that the Taylor Principle (8) is sufficient for determinacy in a model with a rule of the form (79), and that the unique bounded REE of the model assumes the form:

$$\hat{X}_t := \begin{pmatrix}\hat{\pi}_t\\ \hat{y}_t\\ \hat{R}_t\end{pmatrix} = \Omega\hat{X}_{t-1} + shocks, \tag{80}$$

where only the third column of Ω has non-zero entries (i.e. the lagged interest rate is the only endogenous state variable). Agents can in principle learn the REE implemented under the exp MA rule (79) by recursively estimating a PLM that has the same functional form as (80). If the agents learn the REE, they can infer w_c from estimates of the coefficient multiplying $\hat{R}_{t-1}$ in the estimated interest rate rule, $1-w_c$.

Consider first the case of flexible prices. Following Appendix A.7, the model equations are given by

$$\hat{R}_t = \sum_{j=1}^{\infty}\beta^j\left(\beta^{-2}\hat{\pi}^e_{t+j} - \hat{R}^e_{t+j}\right), \tag{81}$$
$$\hat{R}_t = w_c\frac{\psi_p}{\pi^*}\hat{\pi}_t + (1-w_c)\hat{R}_{t-1}, \tag{82}$$

where for brevity there is no random shock. In view of Appendix A.7, it is assumed that agents perceive that output is exogenous ($\hat{y}_t = \hat{y}^e_{t+j} = 0$ for $j \geq 0$). We assume that agents' PLM for $z \in \{\hat{\pi}, \hat{R}\}$ assumes the form:

$$z_t = a_z + b_z\hat{R}_{t-1}.$$

Appendix D.8 gives the proof of the result:

Proposition 9 *Assume that the Taylor principle (8) holds for the exponential moving average rule (82). If prices are fully flexible, then the unique bounded equilibrium is E-stable.*

Next, we consider the case of sticky prices. In this case one needs to revert to numerical analysis, which yields the result:

[51] To derive (78), equation (28) is first modified to allow for output gap targeting.

[52] Note that the equivalence between (78) and (79) breaks down if the ZLB binds in any period.

Remark 6: *Assume that the Taylor principle (8) holds. The exponential moving average rule (79) delivers E-stability and robust stability for many parameter configurations given in Table C.6.*

Table C.6 shows the largest gain parameter consistent with stability of the target equilibrium when agents know the RE slope coefficients in their correctly-specified PLM (Ω in (80)) but estimate the intercept in their PLM using a version of constant-gain learning (similar to Tables II and C.3). It is seen that the target equilibrium is robustly stable under learning under different assumptions about the length of the averaging window (w_c). Hence, agents can learn the target REE and the averaging window by conditioning their forecasts on a lag of the interest rate.

γ	42	128.21	350
ω_0 (IT)	0.04242	0.04545	0.05316
ω_0 $(w_c = 0.5)$	0.03911	0.05179	0.06219
ω_0 $(w_c = 0.2)$	0.03260	0.05437	0.10368
ω_0 $(w_c = 0.01)$	0.02936	0.04069	0.05899

Table C.6: Least upper bounds ω_0 for policy parameters $\psi_p = 1.5$, $\psi_y = 0.125$

Note: the case $w_c = 1$ corresponds to IT.

D Proofs

D.1 Proof of Proposition 1(i)

In the linearization (18)-(19) we get[53]

$$DF_x = \begin{pmatrix} 1 & 0 & \frac{\beta(y^*-g)}{\pi^*\sigma} \\ -\kappa & 1 & 0 \\ -\frac{\psi_y}{y^*} & -\frac{\psi_p}{\pi^*} & 1 \end{pmatrix}$$

$$DF_{x^e} = \begin{pmatrix} -1 & \frac{-(g-y^*)}{\pi^*\sigma(\beta-1)} & \frac{\beta^2(g-y^*)}{\pi^*\sigma(\beta-1)} \\ \frac{\beta}{\beta-1}\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$DF_{x_i} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\psi_p/\pi^* & 0 \end{pmatrix}, \; i = 1, ..., L-1.$$

where

$$\kappa = \frac{\nu\left(\frac{(\nu-1)\sigma y^*(y^*-\bar{g})^{-\sigma-1}}{\nu} - \frac{(\nu-1)(y^*-\bar{g})^{-\sigma}}{\nu} + \frac{(\epsilon+1)y^{*\frac{\epsilon+1}{\alpha}-1}}{\alpha^2}\right)}{\gamma(2\pi^*-1)} \geq 0$$

[53]It can be seen that, with steady state learning, the linearization is the same as in Section 3.1.

if $\sigma > (y^* - \bar{g})/y^*$. It follows that

$$
\begin{aligned}
M &= -(DF_x)^{-1}DF_{x^e} = \\
&\begin{pmatrix}
\frac{y^*\left(\beta^2\kappa\psi_p(y^*-\bar{g})+(\beta-1)\pi*^2\sigma\right)}{\Game} & \frac{\pi^* y^*(\bar{g}-y^*)}{\Game} & \frac{\beta^2\pi^* y^*(y^*-\bar{g})}{\Game} \\
\frac{\kappa(-\pi^*)\left(\beta^2\psi_y(y^*-\bar{g})+\pi^*\sigma y^*\right)}{\Game} & \frac{\kappa\pi^* y^*(\bar{g}-y^*)}{\Game} & \frac{\beta^2\kappa\pi^* y^*(y^*-\bar{g})}{\Game} \\
\frac{\pi^*\sigma((\beta-1)\pi^*\psi_y-\kappa\psi_p y^*)}{\Game} & \frac{(\bar{g}-y^*)(\pi^*\psi_y+\kappa\psi_p y^*)}{\Game} & \frac{\beta^2(y^*-\bar{g})(\pi^*\psi_y+\kappa\psi_p y^*)}{\Game}
\end{pmatrix}, \\
N_i &= -(DF_x)^{-1}DF_{x_i} = \hat{N} = \\
&\begin{pmatrix}
0 & \frac{\beta\psi_p y^*(\bar{g}-y^*)}{\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*} & 0 \\
0 & \frac{\beta\kappa\psi_p y^*(\bar{g}-y^*)}{\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*} & 0 \\
0 & \frac{\pi^*\sigma\psi_p y^*}{\beta\pi^*\psi_y(y^*-\bar{g})+\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*} & 0
\end{pmatrix}, \ i = 1, ..., L-1.
\end{aligned}
$$

where

$$\Game = (\beta - 1)\left(\beta\pi^*\psi_y(y^* - \bar{g}) + \beta\kappa\psi_p y^*(y^* - \bar{g}) + (\pi^*)^2\sigma y^*\right) < 0.$$

Introduce the notation $x_t = (\hat{y}_t, \hat{\pi}_t, \hat{R}_t)$ etc. The linearized the system (18), (19) can be written

$$Z_t = QZ_{t-1}, \text{ where} \tag{83}$$

$$Z_t = \begin{pmatrix} x_t^e & x_t & x_{t-1} & x_{t-2} & \cdots & x_{t-(L-2)} \end{pmatrix}^T$$

$$
Q = \begin{pmatrix}
(1-\omega)I_3 & \omega I_3 & 0 & \cdots & 0 & 0 \\
(1-\omega)M & \omega M + N_1 & N_2 & \cdots & N_{L-2} & N_{L-1} \\
0 & I_3 & 0 & \cdots & 0 & 0 \\
0 & 0 & I_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I_3 & 0
\end{pmatrix}.
$$

For stability, the roots of $P(\lambda) = Det[Q - \lambda I_{3L}]$ must be inside the unit circle. One can show that

$$P(\lambda) = \lambda^{2L-2}(\lambda - (1-\omega))\tilde{P}(\lambda)$$

Thus, the roots of $P(\lambda)$ are inside the unit circle if and only if the roots of $\tilde{P}(\lambda)$ are inside the unit circle. In the limit $\omega \to 0$, we have

$$\tilde{P}(\lambda) = (1-\lambda)^2(\lambda^{L-1} + h\sum_{k=0}^{L-2}\lambda^k)$$

where

$$h = \frac{\beta\kappa\psi_p y^*(y^* - \bar{g})}{\beta\pi^*\psi_y(y^* - \bar{g}) + \beta\kappa\psi_p y^*(y^* - \bar{g}) + (\pi^*)^2\sigma y^*} \in (0, 1)$$

if $\gamma > 0$. Using the stability criterion in Jury (1961), the roots of $(\lambda^{L-1} + h\sum_{k=0}^{L-2}\lambda^k)$ are inside the unit circle if and only if[54]

$$1 - \frac{kh^2}{1 + (k-1)h} > 0, k = 1, \ldots, L,$$

[54] Proof in Mathematica available on request.

which is satisfied for all L. Therefore, the roots of $P(\lambda)$ are inside the unit circle if $\partial\lambda/\partial\omega < 0$ evaluated at $\omega = 0$ and $\lambda = 1$. To evaluate the derivative, we consider the Taylor series expansion of $\tilde{P}(\lambda)$ up to second order at point (λ_0, ω_0). Let $(d\lambda, d\omega) = (\lambda, \omega) - (\lambda_0, \omega_0)$. Then

$$\begin{aligned}\tilde{P}(\lambda,\omega) &= \tilde{P}(\lambda_0,\omega_0) + \tilde{P}_\lambda(\lambda_0,\omega_0)d\lambda + \tilde{P}_\omega(\lambda_0,\omega_0)d\omega + \\ &\quad \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2} + Q,\end{aligned}$$

where subscripts denote partial derivatives and Q is a remainder.

Now

$$\begin{aligned}\tilde{P}_\omega(\lambda_0,\omega_0) &= 0\\ \tilde{P}_\lambda(\lambda_0,\omega_0) &= 0\end{aligned}$$

so we get the approximation

$$\tilde{P}(\lambda,\omega) = \tilde{P}(\lambda_0,\omega_0) + \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2}.$$

Now impose

$$\tilde{P}(\lambda,\omega) - \tilde{P}(\lambda_0,\omega_0) = 0$$

to compute the derivative of the implicit function. So we have

$$\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2} + \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} = 0$$

or

$$\frac{d\lambda}{d\omega} = \frac{-1}{2}\left(\frac{\tilde{P}_{\omega\omega}(\lambda_0,\omega_0)}{\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)} + \frac{\tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)}{\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)}\left(\frac{d\lambda}{d\omega}\right)^2\right)$$

Evaluating the partial derivatives at $(\lambda_0, \omega_0) = (1, 0)$ we have

$$\begin{aligned}\tilde{P}_{\omega\omega}(1,0) &= (-1)^L\frac{2(y^*-\bar{g})((1-\beta)\beta\pi^*\psi_y + \kappa y^*(L\beta\psi_p - \pi^*))}{(\beta-1)^2\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)}\\ \tilde{P}_{\lambda\lambda}(1,0) &= (-1)^L\frac{2\left(\beta\pi^*\psi_y(y^*-\bar{g}) + L\beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)}{\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*}\\ \tilde{P}_{\lambda\omega}(1,0) &= (-1)^L\frac{\kappa y^*(y^*-\bar{g})(\pi^* - 2L\beta\psi_p) + (2-\beta)\beta\pi^*\psi_y(\bar{g}-y^*) - (1-\beta)(\pi^*)^2\sigma y^*}{(\beta-1)\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)}\end{aligned}$$

One can show that $P_{\omega\omega}(1,0) > 0$, $P_{\lambda\lambda}(1,0) > 0$, $P_{\lambda\omega}(1,0) > 0$ if L is even and $P_{\omega\omega}(1,0) < 0$, $P_{\lambda\lambda}(1,0) < 0$, $P_{\lambda\omega}(1,0) < 0$ if L is odd. Therefore, $\partial\lambda/\partial\omega < 0$ and we have stability for small ω and $\kappa > 0$. We note that the same result obtains if $\psi_p > \pi^*/(\beta L)$.

D.2 Proof of Proposition 1(ii)

In the case $\gamma = 0$ the dynamics of output expectations do not depend on the rest of the system and can be shown to be locally convergent. Introducing the notation $\tilde{x}_t = (\hat{\pi}_t, \hat{R}_t)$, the linearization (18)-(19) becomes

$$\begin{aligned}\tilde{M} &\equiv -(D\tilde{F}_x)^{-1}D\tilde{F}_{x^e} = \begin{pmatrix}\frac{\pi^*}{\psi_p\beta(1-\beta)} & -\frac{\beta\pi^*}{\psi_p(1-\beta)}\\ \frac{1}{\beta(1-\beta)} & -\frac{\beta}{(1-\beta)}\end{pmatrix} \text{ and}\\ \tilde{N}_i &\equiv -(D\tilde{F}_x)^{-1}D\tilde{F}_{x_i} = \begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix},\ i = 1, ..., L-1.\end{aligned}$$

The system becomes

$$\tilde{Z}_t = \tilde{Q}\tilde{Z}_{t-1}, \text{ where} \tag{84}$$

$$\tilde{Z}_t = \begin{pmatrix} \tilde{x}_t^e & \tilde{x}_t & \tilde{x}_{t-1} & \tilde{x}_{t-2} & \cdots & \tilde{x}_{t-(L-2)} \end{pmatrix}^T$$

$$\tilde{Q} = \begin{pmatrix} (1-\omega)I_2 & \omega I_2 & 0 & \cdots & 0 & 0 \\ (1-\omega)\tilde{M} & \omega\tilde{M}+\tilde{N}_1 & \tilde{N}_2 & \cdots & \tilde{N}_{L-2} & \tilde{N}_{L-1} \\ 0 & I_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_2 & 0 \end{pmatrix}.$$

Note that in $\tilde{Q}$ we have $\tilde{N}_i = \tilde{N}$ for all i and $\tilde{N}$ is zero except for element $\tilde{n}_{11}$. In the determinant eliminate the second column from each block ≥ 3 and also corresponding row. We get

$$\det[\tilde{Q} - \lambda I_{2L}] = \lambda^{L-2}\det[\tilde{K}_{L+2}], \tag{85}$$

where

$$\tilde{K}_{L+2} = \begin{bmatrix} (1-\omega)I_2 - \lambda I_2 & \omega I_2 & 0 & 0 & \cdots & 0 & 0 \\ (1-\omega)\tilde{M} & \omega\tilde{M}+\tilde{N}_1 - \lambda I_2 & N1 & N1 & \cdots & N1 & N1 \\ 0 & (1,0) & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & -\lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\lambda \end{bmatrix}_{(L+2)\times(L+2)}$$

and $N1 = (-1,0)^T$.

Consider first the case $L = 1$, so there are no lags. We can focus on the learning dynamics of $\hat{\pi}_t$ and $\hat{R}_t$, i.e. the matrix

$$\begin{pmatrix} (1-\omega)I & \omega I \\ (1-\omega)\tilde{M} & \omega\tilde{M} \end{pmatrix}, \text{ where } \tilde{M} = \begin{pmatrix} \frac{-\pi^*}{(\beta-1)\beta\psi_p} & \frac{\beta\pi^*}{(\beta-1)\psi_p} \\ \frac{-1}{(\beta-1)\beta} & \frac{\beta}{\beta-1} \end{pmatrix}.$$

Assume $\psi_p > \beta^{-1}\pi^* = \bar{R}$. When $L = 1$ the system is four dimensional and two of the eigenvalues are those of $\tilde{M}$. Clearly $tr(\tilde{M} - I) < 0$ and $\det(\tilde{M} - I) > 0$. The other two eigenvalues are a repeated root equal to $1 - \omega < 1$ for all small ω. So E-stability holds in this case.

In the case of general L, the characteristic polynomial of K_{L+2} has the following structure:[55]

$$\det[K_{L+2}] = \lambda(\lambda - 1 + \omega)P(n, \omega, \lambda)$$

where $n = L$ and

$$\begin{aligned} P(n,\omega,z) &= z^n + \tilde{b}z^{n-1} + \tilde{c}z^{n-2} + ... + \tilde{c}z + a_n, \text{ where} \\ \tilde{b} &= \frac{\omega}{1-\beta}b_1 \text{ with } b_1 = (1 - \frac{\pi^*}{\beta\psi_p}), \\ \tilde{c} &= \frac{\omega}{1-\beta} \text{ and } a_n = \tilde{c} - 1. \end{aligned} \tag{86}$$

[55]The Mathematica routine is available on request.

Here $b_1, \beta, \omega \in (0,1)$.

To apply the Schur-Cohn conditions the polynomial (86) is written in a general form

$$A(\lambda) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n$$

so

$$a_1 = \tilde{b}, a_2 = \tilde{c}, \ldots, a_{n-1} = \tilde{c} \text{ and } a_n = \tilde{c} - 1. \quad (87)$$

Then define the matrices[56]

$$D_{n-1}^{\pm} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & a_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & \cdots & a_n & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_n & \cdots & a_4 & a_3 \\ a_n & a_{n-1} & \cdots & a_3 & a_2 \end{pmatrix}.$$

The roots of $A(\lambda)$ are inside the unit circle if and only if the following conditions hold: (a) $A(1) > 0$ and (b) $(-1)^n A(-1) > 0$, and (c) the matrices $D_{n-1}^{\pm}$ are positive innerwise, i.e., $|D_1^{\pm}| > 0$, $|D_3^{\pm}| > 0, ..., |D_{n-1}^{\pm}| > 0$ when n is even and $|D_2^{\pm}| > 0$, $|D_4^{\pm}| > 0, ..., |D_{n-1}^{\pm}| > 0$ when n is odd, where $D_i^{\pm} > 0$ denote the inners of $D_{n-1}^{\pm} > 0$ as defined in Elaydi (2005).

Now

$$\begin{aligned} A(1) &= 1 + \tilde{b} + (n-2)\tilde{c} + \tilde{c} - 1 = \frac{\omega}{1-\beta}(b_1 + n - 1) > 0 \text{ if } n \geq 1. \\ (-1)^n A(-1) &= 1 - \tilde{b} + \tilde{c} - 1 = \tilde{c} - \tilde{b} = \frac{\omega}{1-\beta}(1 - b_1) > 0 \text{ if } n \text{ is even and} \\ (-1)^n A(-1) &= (-1)(-1 + \tilde{b} - \tilde{c} + \tilde{c} - 1) = 2 - \tilde{b} > 0 \text{ if } n \text{ is odd,} \end{aligned}$$

so conditions (a) and (b) hold. Substituting in the relations (87) we get

$$D_{n-1}^{\pm} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \tilde{b} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \tilde{c} & \tilde{c} & \cdots & 1 & 0 \\ \tilde{c} & \tilde{c} & \cdots & \tilde{b} & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & \tilde{c}-1 \\ 0 & 0 & \cdots & \tilde{c}-1 & \tilde{c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \tilde{c}-1 & \cdots & \tilde{c} & \tilde{c} \\ \tilde{c}-1 & \tilde{c} & \cdots & \tilde{c} & \tilde{c} \end{pmatrix},$$

where $n \geq 2$. Assume first that n is even and consider the inners $D_1^{\pm}$, $D_3^{\pm}$, ..., $D_{n-1}^{\pm}$. we get

$$|D_1^{\pm}| = 1 \pm (\tilde{c} - 1) = \tilde{c} \text{ or } 2 - \tilde{c} \text{ which are } > 0 \text{ if } \tilde{c} < 2$$

and

$$|D_3^{\pm}| = \left| \begin{pmatrix} 1 & 0 & 0 \\ \tilde{b} & 1 & 0 \\ \tilde{c} & \tilde{b} & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & a_n \\ 0 & a_n & \tilde{c} \\ a_n & \tilde{c} & \tilde{c} \end{pmatrix} \right|$$

yielding $|D_3^{-}| = (\tilde{b} - \tilde{c})(\tilde{c} + \tilde{b} - \tilde{b}\tilde{c}) < 0$ as $0 < \tilde{b} < \tilde{c}$, which implies instability for $n \geq 4$ even, but stability for $n = 2$.

Next assume that n is odd. One computes

$$|D_2^{\pm}| = \left| \begin{pmatrix} 1 & \pm a_n \\ \tilde{b} \pm a_n & 1 \pm \tilde{c} \end{pmatrix} \right|$$

[56] This form of the Schur-Cohn conditions is given in Elaydi (2005), p.247.

so $\left|D_2^+\right| = 3\tilde{c} - b\tilde{c} + \tilde{b} - \tilde{c}^2 > 0$ and $\left|D_2^-\right| = (\tilde{c} - \tilde{b})(1 - \tilde{c}) > 0$ for sufficiently small $\tilde{c} > 0$, which implies stability for $n = 3$. Next consider

$$D_4^{\pm} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tilde{b} & 1 & 0 & 0 \\ \tilde{c} & \tilde{b} & 1 & 0 \\ \tilde{c} & \tilde{c} & \tilde{b} & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & 0 & \tilde{c}-1 \\ 0 & 0 & \tilde{c}-1 & \tilde{c} \\ 0 & \tilde{c}-1 & \tilde{c} & \tilde{c} \\ \tilde{c}-1 & \tilde{c} & \tilde{c} & \tilde{c} \end{pmatrix}.$$

One computes

$$D_4^- = \begin{pmatrix} 1 & 0 & 0 & 1-\tilde{c} \\ \tilde{b} & 1 & 1-\tilde{c} & -\tilde{c} \\ \tilde{c} & 1+\tilde{b}-\tilde{c} & 1-\tilde{c} & -\tilde{c} \\ 1 & 0 & \tilde{b}-\tilde{c} & 1-\tilde{c} \end{pmatrix},$$

so $\left|D_4^-\right| = (\tilde{b} - \tilde{c})^2(\tilde{b}\tilde{c} - \tilde{b} - 1) < 0$ implying instability for n odd. So overall there is instability for $n \geq 4$ and stability for $n \leq 3$.

We further note that $\left|D_5^-\right| = (\tilde{b}-\tilde{c})^2(\tilde{b}^2(1-\tilde{c})+\tilde{c}(1+\tilde{b})-2) < 0$ for small ω. Therefore, we have instability for any $\psi_p > 0$ if $L > 4$, since $\left|D_4^-\right| < 0$ and $\left|D_5^-\right| < 0$.

These last inequalities also hold under the weaker restriction $\psi_p > \pi^*/(\beta L)$.

D.3 Proof of Proposition 2

First consider (i). The model (5), (6) and (7) with i.i.d. government spending shock can be written as

$$\begin{aligned}
\hat{y}_t &= c_R\hat{R}_t + \sum_{j=1}^{\infty}\beta^j\left(c_\pi\hat{\pi}^e_{t+j} + c_y\hat{y}^e_{t+j} + c_R\hat{R}^e_{t+j}\right) + \hat{g}_t \\
\hat{\pi}_t &= \kappa\hat{y}_t + \kappa\sum_{j=1}^{\infty}\beta^j\hat{y}^e_{t+j} + g_{pc}\hat{g}_t \\
\hat{R}_t &= \frac{\psi_p}{\pi^*}\sum_{i=0}^{L-1}\hat{\pi}_{t-i} + \psi_y\frac{\hat{y}_t}{y^*}
\end{aligned}$$

where $\hat{g}_t$ is an i.i.d shock, $c_R = -c^*\beta(\sigma\pi^*)^{-1}$, $c_\pi = -c_R\beta^{-2}$, $c_y = \beta^{-1}(1-\beta)$, g_{pc} is a complicated function of deep parameters, and $\kappa = \kappa(\gamma)$ with $\frac{\partial\kappa}{\partial\gamma} < 0$ and $\lim_{\gamma\to\infty}\kappa = 0$. Suppose the following PLM:

$$\hat{\pi}_t = a_\pi + b_\pi\hat{\pi}_{t-1} \tag{88}$$
$$\hat{R}_t = a_R + b_R\hat{\pi}_{t-1} \tag{89}$$
$$\hat{y}_t = a_y + b_y\hat{\pi}_{t-1} \tag{90}$$

which implies

$$\begin{aligned}
\hat{\pi}^e_{t+j} &= (1-b_\pi^{j+1})a_\pi/(1-b_\pi) + b_\pi^{j+1}\hat{\pi}_{t-1} \\
s^e_{t+j} &= a_s + b_s(1-b_\pi^j)a_\pi/(1-b_\pi) + b_s b_\pi^j\hat{\pi}_{t-1}
\end{aligned}$$

where $s = \hat{y}, \hat{R}$. Under this PLM, a restricted perceptions equilibrium (RPE) of (5), (6) and (7) is given by coefficients $(a_\pi, a_R, a_y, b_\pi, b_R, b_y)$ which satisfy the least-squares orthogonality restriction

$$E\hat{\pi}_{t-1}\left(s_t - a_s - b_s\hat{\pi}_{t-1}\right) = E(\hat{\pi}_{t-1} - a_s)\left(s_t - a_s - b_s\hat{\pi}_{t-1}\right) = 0$$

for $s = \hat{\pi}, \hat{y}, \hat{R}$, where $|b_\pi| < 1$, $b_\pi \neq 0$, and E denotes the unconditional expectations operator. This restriction implies: $a_s = 0$.[57]

Suppose an RPE exists and that agents update their estimates of (a_y, a_R, a_π) according to (22) with b_π, b_R, b_y fixed to their RPE values. Substituting expectations into the system (5), (6) and (7) gives the first-order system:

$$Z_t = QZ_{t-1} + \tilde{\epsilon}_t. \tag{91}$$

where $Z_t = (\hat{\pi}_t, \hat{R}_t, \hat{y}_t, a_{\pi,t}, a_{R,t}, a_{y,t}, \hat{\pi}_{t-1}, \ldots, \hat{\pi}_{t-L+2})^T$. For stability, the roots of $P(\lambda) = Det[Q - \lambda I_{L+4}]$ must be inside the unit circle. One can show that for sufficiently large γ (i.e. in the limit $\kappa \to 0$), $P(\lambda) = -(1 - \frac{c^*\omega\beta\psi_y}{(1-\beta)(y^*\pi^*\sigma + c^*\beta\psi_y)} - \lambda)\lambda^{L+1}(\lambda - (1-\omega))^2$. Hence, we have stability for small κ and small ω if $\psi_y > 0$. To show stability for small κ and small ω when $\psi_y = 0$, we compute the total differential of $P(\lambda)$:

$$\begin{aligned} \frac{\partial P}{\partial \lambda} d\lambda + \frac{\partial P}{\partial \kappa} d\kappa &= 0 \\ \Longrightarrow \frac{d\lambda}{d\kappa} &= -\frac{\partial P}{\partial \kappa} / \frac{\partial P}{\partial \lambda}. \end{aligned}$$

Evaluated at $\lambda = 1$ and $\kappa = \psi_y = 0$ we have

$$\Longrightarrow \frac{d\lambda}{d\kappa} = \frac{c^*\omega(\pi^* - L\beta\psi_p)}{(\beta-1)^2(\pi^*)^2\sigma} < 0$$

Therefore, an RPE is stable under constant gain learning about the intercept term, assuming an RPE exists and κ sufficiently small. We note that same result holds if $\psi_p > \pi^*/(\beta L)$.

Now consider (ii). In the limit $\gamma \to 0$, $\hat{y}_t = g_y \hat{g}_t$ where g_y is a complicated function of deep structural parameters and therefore (5), (6) and (7) reduces to:[58]

$$\hat{R}_t = \sum_{j=1}^{\infty} \beta^j \left(\beta^{-2} \hat{\pi}^e_{t+j} - \hat{R}^e_{t+j} \right) + \epsilon_t, \tag{92}$$

$$\hat{R}_t = \frac{\psi_p}{\pi^*} \sum_{i=0}^{L-1} \hat{\pi}_{t-i}, \tag{93}$$

where ϵ_t is proportional to the $i.i.d$ government spending shock $\tilde{g}_t$, and $\psi_y = 0$ is assumed for simplicity. Suppose the PLM:

$$\hat{\pi}_t = a_\pi + b_\pi \hat{\pi}_{t-1}, \tag{94}$$

$$\hat{R}_t = a_R + b_R \hat{\pi}_{t-1}, \tag{95}$$

which implies

$$\begin{aligned} \hat{\pi}^e_{t+j} &= (1 - b_\pi^{j+1}) a_\pi / (1 - b_\pi) + b_\pi^{j+1} \hat{\pi}_{t-1}, \\ \hat{R}^e_{t+j} &= a_R + b_R (1 - b_\pi^j) a_\pi / (1 - b_\pi) + b_R b_\pi^j \hat{\pi}_{t-1}. \end{aligned}$$

[57] Mathematica routine available on request.

[58] Here, as with other flexible price results, we assume agents learn the exogenous process for output, i.e. $\hat{y}_t = g_y \hat{g}_t$ which implies $\hat{y}^e_{t+j} = 0$ for all $j \geq 1$.

Under the PLM, (94)-(95), a restricted perceptions equilibrium (RPE) of (92)-(93) is given by coefficients (a_π, a_R, b_π, b_R) which satisfy the least-squares orthogonality restriction

$$\begin{aligned} E\hat{\pi}_{t-1}\left(\pi_t - a_\pi - b_\pi \hat{\pi}_{t-1}\right) = E(\hat{\pi}_{t-1} - a_\pi)\left(\hat{\pi}_t - a_\pi - b_\pi \hat{\pi}_{t-1}\right) &= 0, \\ E\hat{\pi}_{t-1}\left(\hat{R}_t - a_R - b_R \hat{\pi}_{t-1}\right) = E(\hat{\pi}_{t-1} - a_R)\left(\hat{R}_t - a_R - b_R \hat{\pi}_{t-1}\right) &= 0, \end{aligned}$$

where $|b_\pi| < 1$, $b_\pi \neq 0$, and E denotes the unconditional expectations operator. This restriction implies: $a_R = a_\pi = 0$, $b_R = \frac{b_\pi^2}{\beta}$.[59]

Suppose an RPE exists and that agents update their estimates of (a_R, a_π) according to (22) with b_π and b_R fixed to their RPE values. Substituting expectations (22) and (93) into (92) yields:

$$\begin{aligned} \hat{\pi}_t &= d_\pi a_{\pi,t} + d_R a_{R,t} + e_\pi \hat{\pi}_{t-1} - \sum_{i=2}^{L-1} \hat{\pi}_{t-i} + \tilde{\epsilon}_t \\ &= (e_\pi + d_\pi \omega)\hat{\pi}_{t-1} + (-d_\pi b_\pi \omega - d_R b_R \omega - 1)\hat{\pi}_{t-2} \\ &+ d_R \omega \hat{R}_{t-1} + d_\pi (1-\omega) a_{\pi,t-1} + d_R(1-\omega) a_{R,t-1} - \sum_{i=3}^{L-1} \hat{\pi}_{t-i} + \tilde{\epsilon}_t, \end{aligned} \tag{96}$$

where

$$\begin{aligned} e_\pi &= \frac{\pi^* b_\pi (b_\pi \beta^{-1} - b_R \beta)}{\psi_p (1 - \beta b_\pi)} - 1, \\ d_\pi &= \frac{\pi^*}{\psi_p (1 - b_\pi)} \left(\frac{\beta^{-1} - \beta b_R}{1 - \beta} - \frac{b_\pi \left(\beta^{-1} b_\pi - \beta b_R\right)}{1 - \beta b_\pi} \right), \\ d_R &= -\frac{\pi^* \beta}{\psi_p (1 - \beta)}. \end{aligned}$$

Introduce the notation $Z_t = (\hat{\pi}_t, \hat{R}_t, a_{\pi,t}, a_{R,t}, \hat{\pi}_{t-1}, \ldots, \hat{\pi}_{t-L+2})^T$. Modifying the system gives

$$Z_t = Q Z_{t-1} + \hat{\epsilon}_t. \tag{97}$$

For stability, the roots of $P(\lambda) = Det[Q - \lambda I_{L+2}]$ must be inside the unit circle. One can show that in the limit $\omega \to 0$

$$P(\lambda) = -(1-\lambda)^2 \lambda \tilde{P}(\lambda)$$

Thus, some roots of $P(\lambda)$ are outside of the unit circle if any root of $\tilde{P}(\lambda)$ is outside the unit circle where

$$\tilde{P}(\lambda) = \lambda^{L-1} + \left(1 - \frac{\pi^* b_\pi^2}{\beta \psi_p}\right) \lambda^{L-2} + \sum_{k=0}^{L-3} \lambda^k \tag{98}$$

where $0 < \frac{\pi^* b_\pi^2}{\beta \psi_p} < 1$ under the Taylor Principle with $\pi^* \geq 1$. The preceding equation has the following form

$$Q(\lambda) = \lambda^n + (1-c)\lambda^n + \sum_{k=0}^{n-2} \lambda^k$$

[59]Mathematica routine available on request. Given $a_R = a_\pi = 0$, we can show that $\hat{R}_t = B(b_R)\hat{\pi}_{t-1} + r\epsilon_t$ where r is a scalar, $B(b_R) = \frac{\psi_p}{\pi^*}(e_\pi + 1)$ and e_π is defined in (96) below. Therefore, $\frac{E(\hat{R}_t \hat{\pi}_{t-1})}{E(\hat{\pi}_t \hat{\pi}_t)} = B(b_R) = b_R$ since $a_R = 0$. Solving $B(b_R) = b_R$ for b_R gives $b_R = \frac{b_\pi^2}{\beta}$.

where $c \in (0,1)$ and $n = L-1$. We can assess stability following the Schur-Cohn conditions presented in the proof of Proposition 1(ii).

If n is even, then

$$\begin{aligned} \left|D_3^{\pm}\right| &= \left| \begin{pmatrix} 1 & 0 & 0 \\ 1-c & 1 & 0 \\ 1 & 1-c & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right| \\ & -2c + c^2 \text{ or } -c^2 \end{aligned}$$

so $\left|D_3^+\right| < 0$ and $\left|D_3^-\right| < 0$, which implies instability.

Then consider the case n is odd.

$$\left|D_2^{\pm}\right| = \left| \begin{pmatrix} 1 & \pm 1 \\ 1-c\pm 1 & 1\pm 1 \end{pmatrix} \right| = (1\pm 1) \mp (1-c\pm 1) = c \text{ or } -c.$$

So there is instability as $\left|D_2^-\right| < 0$. So overall there is instability for $L \geq 4$. We note that the result holds for any $\psi_p > 0$. In particular, the proposition holds with the weaker restriction $\psi_p > \pi^*/(\beta L)$.

D.4 Proof of Proposition 3

It is seen that only lags of inflation appear in structural model (76). Its coefficient matrices $\mathcal{A}_i$ take the form given in (77), and the resulting temporary equilibrium is $VAR(L-1)$. Let $z_t = (\hat{R}_t, \hat{\pi}_t)^T$. The temporary equilibrium system (unprojected ALM) takes the form

$$z_t = \mathcal{A}_1 z_{t-1} + ... + \mathcal{A}_{N-1} z_{t-(N-1)} + \hat{N} z_{t-N} + ... + \hat{N} z_{t-(L-1)} + w_t, \tag{99}$$

where w_t is iid random shock. The relevant characteristic polynomial is $\tilde{H}(\lambda) = \lambda^{L-1} H(\lambda)$, where

$$H(\lambda) = \lambda^{L-1} - \sum_{i=1}^{N-1} a_{i22} \lambda^{L-1-i} + \sum_{i=N}^{L-1} \lambda^{L-1-i},$$

The temporary equilibrium system is stationary if the roots of $H(\lambda)$ are inside the unit circle. The Schur-Cohn conditions are the relevant stability conditions. According to Proposition 5.1 in Elaydi (2005), condition (iii) is necessary for $H(\lambda)$ to have all of its roots inside the unit circle. This condition is stated in terms of the inners of the following matrices

$$B_{L-2}^{\pm} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a_{122} & 1 & 0 & \cdots & 0 \\ -a_{222} & -a_{122} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{L-3} & b_{L-4} & \cdots & -a_{122} & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & b_{L-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & b_4 & b_3 \\ 1 & b_{L-2} & \cdots & b_3 & -a_{222} \end{pmatrix}.$$

where $b_k = -a_{k22}$ if $N > k$ or where $b = 1$ if $N \leq k$. The smallest inner of $B_{L-2}^{\pm}$ is either

$$\left|B_1^{\pm}\right| = 1 \pm 1 = 2 \text{ or } 0$$

if L is odd or

$$\begin{aligned} \left|B_2^{\pm}\right| &= \left|\begin{pmatrix} 1 & 0 \\ -a_{122} & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 1 \\ 1 & b_{L-2} \end{pmatrix}\right| \\ &= \left|\begin{pmatrix} 1 & 1 \\ 1-a_{122} & 1+b_{L-2} \end{pmatrix} \text{ or } \begin{pmatrix} 1 & -1 \\ -a_{122}-1 & 1-b_{L-2} \end{pmatrix}\right| \\ &= b_{L-2}+a_{122} \text{ or } -(b_{L-2}+a_{122}). \end{aligned}$$

where $b_{L-2} = -a_{(L-2)22}$ or -1 if L is even. So there is always a zero inner, which implies that not all roots are inside the unit circle.[60]

D.5 Proofs of Lemma 4 and Proposition 5

D.5.1 Proof of Lemma 4

We continue from the beginning of Section 4, so introduce the notation for the PLM:

$$\tilde{X}_{R,t} = \tilde{A}_{R,0} + \tilde{A}_R \tilde{X}_{R,t-1}, \tag{100}$$

where

$$\tilde{A}_R = \begin{pmatrix} A_1 & A_2 & \cdots & A_{R-1} \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & I & 0 \end{pmatrix} \text{ and } \tilde{A}_{R,0} = \begin{pmatrix} A_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Note that in (23) the number of lags that arise structurally is $L-1$, which is smaller than $R-1$ so the additional lags arise purely in the PLM. Stacking the system into first order form gives the temporary equilibrium system of equations

$$\tilde{X}_{R,t} = \tilde{K} + \sum_{i=1}^{\infty} \beta^i \tilde{M}_R \tilde{X}^e_{R,t,t+i} + \tilde{N}_2 \tilde{X}_{R,t-1} + Z_t, \tag{101}$$

where Z_t is a vector of $i.i.d.$ shocks. The system is written out as

$$\begin{aligned} \begin{pmatrix} \hat{X}_t \\ \hat{X}_{t-1} \\ \vdots \\ \hat{X}_{t-(R-2)} \end{pmatrix} &= \begin{pmatrix} K \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sum_{i=1}^{\infty} \begin{pmatrix} \beta^i M & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \hat{X}^e_{t,t+i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &\quad + \tilde{N}_2 \begin{pmatrix} \hat{X}_{t-1} \\ \hat{X}_{t-2} \\ \vdots \\ \hat{X}_{t-(R-1)} \end{pmatrix} + Z_t, \end{aligned}$$

where the matrix

$$\tilde{N}_2 = \begin{pmatrix} \hat{N} & \cdots & \hat{N} & 0 & \cdots & 0 \\ I & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \cdots & \vdots & \vdots \\ 0 & \cdots & I & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & I & 0 \end{pmatrix}$$

[60] It is not necessary to consider E-stability as the unprojected ALM is not stationary.

where the lags of the AIT policy rule are in the top left corner of $\tilde{N}_2$: the top left has $L-1$ matrices $\hat{N}$.

Iterating the PLM (100):

$$\tilde{X}^e_{R,t,t+i} = (I + \tilde{A}_R + ... + \tilde{A}^i_R)\tilde{A}_{R,0} + \tilde{A}^{i+1}_R \tilde{X}_{R,t-1}.$$

After substituting the PLM into the ALM (101), the temporary equilibrium mapping PLM→ALM can be simplified to

$$\tilde{A}_R \rightarrow \sum_{i=1}^{\infty} \beta^i \tilde{M}_R \tilde{A}^{i+1}_R + \tilde{N}_2, \tag{102}$$

$$\tilde{A}_{R,0} \rightarrow \sum_{i=1}^{\infty} \beta^i \tilde{M}_R (I + \tilde{A}_R + ... + \tilde{A}^i_R)\tilde{A}_{R,0} + \tilde{K}. \tag{103}$$

Assuming that the eigenvalues of $\tilde{A}_R$ and $\beta\tilde{A}_R$ are inside the unit circle, the mapping PLM→ALM simplifies to

$$\tilde{A}_R \rightarrow \beta\tilde{M}_R \tilde{A}^2_R \left(I - \beta\tilde{A}_R\right)^{-1} + \tilde{N}_2 \tag{104}$$

$$\tilde{A}_{R,0} \rightarrow \tilde{K} + \beta\tilde{M}_R \left(I - \tilde{A}_R\right)^{-1} \left((1-\beta)^{-1} I - \tilde{A}^2_R \left(I - \beta\tilde{A}_R\right)^{-1}\right) \tilde{A}_{R,0}, \tag{105}$$

where $\tilde{N}_2$ is as above. One computes

$$\beta\tilde{M}_R\tilde{A}^2_R = \begin{pmatrix} \beta M & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} A_1^2 + A_2 & A_1A_2 + A_3 & \cdots & \cdots & A_1A_{R-2} + A_{R-1} & A_1A_{R-1} \\ A_1 & A_2 & \cdots & \cdots & A_{R-2} & A_{R-1} \\ I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & I & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \beta M(A_1^2 + A_2) & \beta M(A_1A_2 + A_3) & \cdots & \beta M(A_1A_{R-2} + A_{R-1)} & \beta M A_1 A_{R-1} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

For mapping (104) define

$$F(\tilde{A}_R) = \beta\tilde{M}_R\tilde{A}^2_R(I - \beta\tilde{A}_R)^{-1}, \tag{106}$$

which is $3(R-1) \times 3(R-1)$ matrix.

Next, consider equation (106) and the REE equation from (104): $F(\tilde{A}_R) + \tilde{N}_2 = \tilde{A}_R$. Multiplying both sides by $(I - \beta\tilde{A}_R)$ we get the equation

$$\beta\bar{M}_R\tilde{A}^2_R = (\tilde{A}_R - \tilde{N}_2)(I - \beta\tilde{A}_R) \tag{107}$$

which can be written as

$$\begin{pmatrix} \beta M(A_1^2+A_2) & \beta M(A_1A_2+A_3) & \cdots & \beta M(A_1A_{R-2}+A_{R-1}) & \beta MA_1A_{R-1} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A_1-\hat{N} & A_2-\hat{N} & \cdots & A_{L-1}-\hat{N} & A_L & \cdots & A_{R-1} \\ 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \ddots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \end{pmatrix}$$

$$* \begin{pmatrix} I-\beta A_1 & -\beta A_2 & \cdots & -\beta A_{L-1} & \cdots & -\beta A_{R-2} & -\beta A_{R-1} \\ -\beta I & I & \cdots & 0 & \cdots & 0 & 0 \\ 0 & -\beta I & \ddots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \ddots & I & 0 \\ 0 & 0 & \cdots & 0 & \cdots & -\beta I & I \end{pmatrix}.$$

We get the equations for the REE

$$\begin{aligned} \beta(M+I)A_1^2-(I+\beta\hat{N})A_1+\hat{N} &= -\beta(M+I)A_2+\beta\hat{N} \\ [\beta(M+I)A_1-(I+\beta\hat{N})]A_2+\hat{N} &= -\beta(M+I)A_3+\beta\hat{N} \\ &\vdots \\ [\beta(M+I)A_1-(I+\beta\hat{N})]A_{L-1}+\hat{N} &= -\beta(I+M)A_L \\ [\beta(M+I)A_1-(I+\beta\hat{N})]A_L &= -\beta(I+M)A_{L+1} \\ &\vdots \\ [\beta(M+I)A_1-(I+\beta\hat{N})]A_{R-2} &= -\beta(M+I)A_{R-1} \\ [\beta(M+I)A_1-(I+\beta\hat{N})]A_{R-1} &= 0. \end{aligned} \tag{108}$$

Generically, for values of A_1 in the last line of (108) the matrix $\beta(M+I)A_1-(I+\beta\hat{N})$ is invertible. It follows that $A_{R-1}=0$ and

$$[\beta(M+I)A_1-(I+\beta\hat{N})]A_{R-2}=0.$$

Continuing this way we get

$$A_L=...=A_{R-1}=0$$

so the fixed point in learning with overparameterized PLM is an MSV solution.

We note that the first equation of (108) yields

$$A_2=-A_1^2+\beta^{-1}(M+I)^{-1}(I+\beta\hat{N})A_1-(M+I)^{-1}(\beta^{-1}-1)\hat{N}, \tag{109}$$

so A_2 is quadratic in A_1 and A_3 is linear in A_2 and in the product $A_1 A_2$ etc. Thus, in general there can be multiple solutions that take the MSV form.

The same argument can be applied in the case of flexible prices.

D.5.2 Proof of Proposition 5:

Denote the REE as $(\tilde{A}_R, \tilde{A}_{R,0}) = (\bar{A}_R, \bar{A}_0)$. The E-stability conditions take the usual form (see (63)) with $R-1$ lags in the PLM

$$\begin{aligned}
DT(\bar{A}_R) &= \left(\left(I-\beta\bar{A}_R\right)^{-1}\beta\bar{A}_R^2\right)^T \otimes \left(\tilde{M}_R\left(I-\beta\bar{A}_R\right)^{-1}\beta\right) + \\
&\quad I \otimes \left(\tilde{M}_R\left(I-\beta\bar{A}_R\right)^{-1}\beta\bar{A}_R\right) + \bar{A}_R^T \otimes \left(\tilde{M}_R\left(I-\beta\bar{A}_R\right)^{-1}\beta\right) \\
&= \left(\left(I-\beta\bar{A}_R\right)^{-1}\beta\bar{A}_R^2 + \bar{A}_R\right)^T \otimes \left(\tilde{M}_R\left(I-\beta\bar{A}_R\right)^{-1}\beta\right) \\
&\quad + I \otimes \left(\tilde{M}_R\left(I-\beta\bar{A}_R\right)^{-1}\beta\bar{A}_R\right) \\
DT(\bar{A}_0) &= \beta\tilde{M}_R\left(I-\bar{A}_R\right)^{-1}\left((1-\beta)^{-1}I - \bar{A}_R^2\left(I-\beta\bar{A}_R\right)^{-1}\right),
\end{aligned}$$

and these are evaluated at the MSV solution with the overparametrization parameters equal to zero. Thus the key matrices are now

$$\begin{aligned}
\tilde{A}_R &= \bar{A}_R = \begin{pmatrix} \bar{A}_1 & \cdots & \bar{A}_{L-1} & \cdots & 0 \\ I & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & \cdots & I & \cdots & 0 \\ 0 & \cdots & \cdots & I & 0 \end{pmatrix} \text{ and} \\
I-\beta\bar{A}_R &= \begin{pmatrix} I-\beta\bar{A}_1 & -\beta\bar{A}_2 & \cdots & -\beta\bar{A}_{L-1} & \cdots & 0 \\ -\beta I & I & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\beta I & I & 0 \\ 0 & 0 & \cdots & 0 & -\beta I & I \end{pmatrix}.
\end{aligned}$$

We partition $\bar{A}_R$ and $I-\beta\bar{A}_R$ so that A_{LL} and D_{LL} contain the corresponding model without overparameterization.

$$\begin{aligned}
\bar{A}_R &= \begin{pmatrix} A_{LL} & 0 \\ B_L & C_L \end{pmatrix}, \text{ where } A_{LL} = \begin{pmatrix} \bar{A}_1 & \bar{A}_2 & \cdots & \bar{A}_{L-1} \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix} \\
B_L &= \begin{pmatrix} 0 & \cdots & I \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, C_L = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ I & \cdots & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix} \text{ and so } \bar{A}_R^2 = \begin{pmatrix} A_{LL}^2 & 0 \\ B_L A_{LL} + C_L B_L & C_L^2 \end{pmatrix}
\end{aligned}$$

and

$$I - \beta \bar{A}_R = \begin{pmatrix} D_{LL} & 0 \\ D_L & E_L \end{pmatrix}, \text{ where } D_{LL} = \begin{pmatrix} I - \beta \bar{A}_1 & -\beta \bar{A}_2 & \cdots & -\beta \bar{A}_{L-1} \\ -\beta I & I & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\beta I & I \end{pmatrix}$$

$$D_L = -\beta B_L \text{ and } E_L = \begin{pmatrix} I & \cdots & \cdots & 0 \\ -\beta I & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\beta I & I \end{pmatrix}.$$

Here A_{LL} and D_{LL} are $3(L-1) \times 3(L-1)$ submatrices of $\bar{A}_R$ and $I - \beta \bar{A}_R$ (alternatively, $2(L-1) \times 2(L-1)$ submatrices under flexible prices).

Using the rule for inverse of partitioned matrices we have

$$\left(I - \beta \bar{A}_R\right)^{-1} = \begin{pmatrix} D_{LL}^{-1} & 0 \\ -E_L^{-1} D_L D_{LL}^{-1} & E_L^{-1} \end{pmatrix}.$$

Consider the first stability condition is that the eigenvalues of $DT(\tilde{A}_R)$ have real parts less than 1. Computing the different terms in $DT(\tilde{A}_R)$, one obtains

$$\begin{aligned} & DT(\tilde{A}_R) \\ = & \begin{pmatrix} ((\beta D_{LL}^{-1} A_{LL} + I)(A_{LL}))^T & \beta[-(E_L^{-1} D_L D_{LL}^{-1} A_{LL}^2)^T + (E_L^{-1}(B_L A_{LL} + C_L B_L))^T] + B_L^T \\ 0 & \beta(E_L^{-1} C_L^2 + C_L)^T \end{pmatrix} \\ & \otimes \begin{pmatrix} \beta \tilde{M} D_{LL}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + I \otimes \begin{pmatrix} \beta \tilde{M} D_{LL}^{-1} A_{LL} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Introducing the matrix notations $\beta \tilde{M} D_{LL}^{-1} = (\beta m_{ij})$, $\beta \tilde{M} D_{LL}^{-1} A_{LL} = (\beta n_{ij})$, $((\beta D_{LL}^{-1} A_{LL} + I)(A_{LL}))^T = (l_{ij})$ and $(\beta E_L^{-1} C_L^2 + C_L)^T = (s_{ij})$, it can be shown that $DT(\tilde{A}_R)$ is block triangular with diagonal blocks

$$BB_1 = \begin{pmatrix} \begin{pmatrix} l_{11}(\beta m_{ij}) + (\beta n_{ij}) & 0 \\ 0 & 0 \end{pmatrix} & \cdots & \begin{pmatrix} l_{1k}(\beta m_{ij}) & 0 \\ 0 & 0 \end{pmatrix} \\ \vdots & & \vdots \\ \begin{pmatrix} l_{k1}(\beta m_{ij}) & 0 \\ 0 & 0 \end{pmatrix} & \cdots & \begin{pmatrix} l_{kk}(\beta m_{ij}) + (\beta n_{ij}) & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

and

$$BB_2 = \begin{pmatrix} \begin{pmatrix} s_{11}(\beta m_{ij}) + (\beta n_{ij}) & 0 \\ 0 & 0 \end{pmatrix} & \cdots & \begin{pmatrix} s_{1q}(\beta m_{ij}) & 0 \\ 0 & 0 \end{pmatrix} \\ \vdots & & \vdots \\ \begin{pmatrix} s_{q1}(\beta m_{ij}) & 0 \\ 0 & 0 \end{pmatrix} & \cdots & \begin{pmatrix} s_{qq}(\beta m_{ij}) + (\beta n_{ij}) & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

The eigenvalues of the block BB_2 consist of zeroes and roots of the equation

$$|BB_2 - \lambda I| = \begin{vmatrix} (s_{11}(\beta m_{ij}) + (\beta n_{ij}) - \lambda I) & \cdots & (s_{1q}(\beta m_{ij})) \\ \vdots & & \vdots \\ (s_{q1}(\beta m_{ij})) & \cdots & (s_{qq}(\beta m_{ij}) + (\beta n_{ij}) - \lambda I) \end{vmatrix}.$$

and to compute the eigenvalues of BB_1, we have

$$|BB_1 - \lambda I| = \begin{vmatrix} (l_{11}(\beta m_{ij}) + (\beta n_{ij}) - \lambda I) & \cdots & (l_{1k}(\beta m_{ij})) \\ \vdots & & \vdots \\ (l_{k1}(\beta m_{ij})) & \cdots & (l_{kk}(\beta m_{ij}) + (\beta n_{ij}) - \lambda I) \end{vmatrix}.$$

It can be noted that the matrices (l_{ij}), (m_{ij}) and (n_{ij}) depend only on matrices A_{LL} and D_{LL}, whereas (s_{ij}) depends on the matrices A_L, B_L, D_L and E_L. In the case of correctly specified PLM

$$DT(\tilde{A}_R) = BB_1 = \left(((\beta D_{LL}^{-1} A_{LL} + I)(A_{LL}))^T\right) \otimes \left(\beta \tilde{M} D_{LL}^{-1}\right) + I \otimes \left(\beta \tilde{M} D_{LL}^{-1} A_{LL}\right)$$

so the weak E-stability condition is that the roots of equation $|BB_1 - \lambda I| = 0$ have real parts less than one. For strong E-stability there is an additional condition that the roots of $|BB_2 - \lambda I| = 0$ must have real parts less than one. It is important to note that the condition for strong E-stability depends on the existence of high order lags of the PLM.[61]

Then consider the E-stability condition for the intercept, i.e. $DT(\bar{A}_0)$ evaluated at $\bar{A}_0, \bar{A}_R$

$$DT(\bar{A}_0) = \beta \tilde{M}_R \left(I - \bar{A}_R\right)^{-1} \left((1-\beta)^{-1} I - \bar{A}_R^2 \left(I - \beta \bar{A}_R\right)^{-1}\right),$$

so the E-stability condition is that the eigenvalues of matrix

$$DT(\bar{A}_0) = \begin{pmatrix} \beta \tilde{M}(I - A_{LL})^{-1}((1-\beta)^{-1} I - A_{LL}^2 D_{LL}^{-1}) & 0 \\ 0 & 0 \end{pmatrix}$$

have real part less than 1. It is seen that this is just the weak E-stability condition for the intercept.

D.6 Proof of Proposition 7

The dynamic model is given by a linearized system of the form (18) and (19) where the interest rate is now (27). In this proof, we set $\psi_p = \psi_p / (\sum_{i=0}^{L-1} \mu^i)$ (i.e. we write the averaging constant explicitly in the interest rate rule, but this detail is not essential for the results). Again in the limit $\gamma \to 0$ the first equation is independent from the rest of the system and output expectations $\hat{y}_t^e$ are convergent. Separating the equation for $\hat{y}_t^e$, the state variables are $\tilde{x}_t = (\hat{\pi}_t, \hat{R}_t)^T$ and the linearized system is of the form (84) but the coefficient matrices $\tilde{M}$ and $\tilde{N}_i$ change to

$$\begin{aligned} \tilde{M} &= -(D\tilde{F}_x)^{-1} D\tilde{F}_{x^e} = \begin{pmatrix} \frac{\pi^* \left(\sum_{i=0}^{L-1} \mu^i\right)}{\psi_p \beta (1-\beta)} & -\frac{\beta \pi^* \left(\sum_{i=0}^{L-1} \mu^i\right)}{\psi_p (1-\beta)} \\ \frac{-1}{(\beta-1)\beta} & \frac{\beta}{\beta-1} \end{pmatrix}, \\ \tilde{N}_i &= -(D\tilde{F}_x)^{-1} D\tilde{F}_{x_i} = \begin{pmatrix} -\mu^i & 0 \\ 0 & 0 \end{pmatrix}, \ i = 1, ..., L-1. \end{aligned}$$

and the system is now

$$\tilde{Z}_t = \tilde{Q}_2 \tilde{Z}_{t-1}, \tag{110}$$

[61] This feature is in line with the analysis of conditions for weak and strong E-stability for simple models discussed in Evans and Honkapohja (2001), chapter 9.

where $\tilde{Z}_t$ is defined in the proof of Proposition 1 (ii), but $\tilde{Q}_2$ incorporates the new forms of $\tilde{M}$ and $\tilde{N}_i$ in $\tilde{Q}$. Consider the characteristic polynomial of $\tilde{Q}_2$ of (110)

$$\det[\tilde{Q}_2 - \lambda I_{2L}] = 0. \tag{111}$$

Given that the second columns of $\tilde{N}_i$ are zero vectors, the determinant in (111) has $L-2$ roots equal to zero. Then analyzing the remaining $(L+2)$ dimensional determinant, again it turns out that there is one more zero root and one root equal to $1-\omega$. Factoring out these, we are left with a polynomial of degree L. Introducing more familiar notation $n = L$, the polynomial is

$$P_2(n,\omega,\lambda) = \lambda^n + b(\omega)\lambda^{n-1} + a(\omega)[\mu\lambda^{n-2}... + \mu^{n-2}\lambda] + (a(\omega)\mu^{n-1} - \mu^n), \tag{112}$$

where μ is the weight parameter in (27),

$$a(\omega) = \frac{\omega}{1-\beta} + \mu - 1, b(\omega) = a(\omega) - \frac{\omega}{1-\beta}b_1 \text{ with } b_1 = \frac{\pi^* \left(\sum_{i=0}^{n-1} \mu^i\right)}{\beta\psi_p}$$

and where $\pi^* < \beta\psi_p$ and $n \geq 2$ are assumed. We again consider how any root varies as ω varies from 0 to small values $d\omega > 0$ and require that in this variation the root is continuously a root of the characteristic polynomial. If $\omega \to 0$ we have $a(\omega) \to \mu - 1$ and $b(\omega) \to \mu - 1$, so the characteristic equation becomes

$$(1-\lambda)(\lambda^{n-1} + \mu\lambda^{n-2} + \mu^2\lambda^{n-3} + ... + \mu^{n-2}\lambda + \mu^{n-1}). \tag{113}$$

There is one root of unity. For the other roots one can apply a generalization of the classic Enerström-Kakeya theorem in Gardner and Govil (2014), Theorem 3.6, stating that the other roots of the polynomial in (113) satisfy $|\lambda| < \mu < 1$.

Then consider the root of 1. Assume now a small perturbation $\omega > 0$. By continuity of eigenvalues the $n-1$ roots that are approximate to the roots of the latter polynomial in (113) remain inside the unit circle. To determine whether the unit root contributes to stability we compute the partial derivatives

$$\begin{aligned}
\frac{\partial P_2}{\partial \lambda} &= n\lambda^{n-1} + (n-1)b(\omega)\lambda^{n-2} + a(\omega)[\mu(n-2)\lambda^{n-3}... + \mu^{n-2}], \\
\frac{\partial P_2}{\partial \omega} &= +b'(\omega)\lambda^{n-1} + a'(\omega)[\mu\lambda^{n-2} + ... + \mu^{n-2}\lambda] + a'(\omega)\mu^{n-1}.
\end{aligned}$$

At $\omega = 0$ and $\lambda = 1$ we have

$$\begin{aligned}
\frac{\partial P_2}{\partial \lambda} &= \frac{1-\mu^n}{1-\mu} > 0, \\
\frac{\partial P_2}{\partial \omega} &= \frac{1}{1-\beta}\left(1 - \frac{\pi^* \sum_{k=0}^{n-1} \mu^k}{\beta\psi_p} + \mu\frac{1-\mu^{n-1}}{1-\mu}\right) > 0,
\end{aligned}$$

since $a'(0) = (1-\beta)^{-1}$ and $b'(0) = (1-\beta)^{-1}(1-b_1)$. Then taking the differential of (112) and requiring

$$\frac{\partial P_2}{\partial \omega}d\omega + \frac{\partial P_2}{\partial \lambda}d\lambda = 0 \implies \frac{\partial \lambda}{\partial \omega} < 0.$$

So for small $\omega > 0$ the real root corresponding to limit 1 is inside the unit circle.

Next consider the case: $\gamma > 0$. In the linearization we get

$$DF_x = \begin{pmatrix} 1 & 0 & \frac{\beta(y^*-g)}{\pi^*\sigma} \\ -\kappa & 1 & 0 \\ -\frac{\psi_y}{y^*} & -\frac{\psi_p}{\pi^*\left(\sum_{i=0}^{L-1}\mu^i\right)} & 1 \end{pmatrix}$$

$$DF_{x^e} = \begin{pmatrix} -1 & \frac{-(g-y^*)}{\pi^*\sigma(\beta-1)} & \frac{\beta^2(g-y^*)}{\pi^*\sigma(\beta-1)} \\ \frac{\beta}{\beta-1}\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$DF_{x_{-i}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{\mu^i\psi_p}{\pi^*\left(\sum_{i=0}^{L-1}\mu^i\right)} & 0 \end{pmatrix}, \; i = 1, ..., L-1,$$

where

$$\kappa = \frac{\nu\left(\frac{(\nu-1)\sigma y^*(y^*-\bar{g})^{-\sigma-1}}{\nu} - \frac{(\nu-1)(y^*-\bar{g})^{-\sigma}}{\nu} + \frac{(\epsilon+1)y^{*\frac{\epsilon+1}{\alpha}-1}}{\alpha^2}\right)}{\gamma(2\pi^*-1)} \geq 0.$$

It follows that

$$M = -(DF_x)^{-1}DF_{x^e} =$$

$$\begin{pmatrix} \frac{y^*\left(\beta^2\kappa\psi_p(y^*-\bar{g})/\left(\sum_{i=0}^{L-1}\mu^i\right)+(\beta-1)(\pi^*)^2\sigma\right)}{\eth_2} & \frac{\pi^*y^*(\bar{g}-y^*)}{\eth_2} & \frac{\beta^2\pi^*y^*(y^*-\bar{g})}{\eth_2} \\ \frac{\kappa(-\pi^*)\left(\beta^2\psi_y(y^*-\bar{g})+\pi^*\sigma y^*\right)}{\eth_2} & \frac{\kappa\pi^*y^*(\bar{g}-y^*)}{\eth_2} & \frac{\beta^2\kappa\pi^*y^*(y^*-\bar{g})}{\eth_2} \\ \frac{\pi^*\sigma((\beta-1)\pi^*\psi_y-\kappa\psi_p y^*/\left(\sum_{i=0}^{L-1}\mu^i\right))}{\eth_2} & \frac{(\bar{g}-y^*)(\pi^*\psi_y+\kappa\psi_p y^*)}{\eth_2} & \frac{\beta^2(y^*-\bar{g})(\pi^*\psi_y+\kappa\psi_p y^*)}{\eth_2} \end{pmatrix},$$

$$N_i = -(DF_x)^{-1}DF_{x_i} =$$

$$\begin{pmatrix} 0 & \frac{\mu^i\beta\psi_p y^*(\bar{g}-y^*)(\beta-1)}{\left(\sum_{i=0}^{L-1}\mu^i\right)a} & 0 \\ 0 & \frac{\mu^i\beta\kappa\psi_p y^*(\bar{g}-y^*)(\beta-1)}{\left(\sum_{i=0}^{L-1}\mu^i\right)a} & 0 \\ 0 & \frac{\mu^i\pi^*\sigma\psi_p y^*(\beta-1)}{\left(\sum_{i=0}^{L-1}\mu^i\right)a} & 0 \end{pmatrix}, \; i = 1, ..., L-1.$$

where

$$\eth_2 = (\beta-1)\left(\pi^*(\beta\psi_y(y^*-\bar{g})+\pi^*\sigma y^*) + \beta\kappa\psi_p y^*(y^*-\bar{g})/\left(\sum_{i=0}^{L-1}\mu^i\right)\right) < 0.$$

The system is now like (83)

$$Z_t = Q_2 Z_{t-1},$$

where Z_t is as before in Proposition 1, but Q_2 incorporates the new forms of M and N_i. Introduce the notation $x_t = (\hat{y}_t, \hat{\pi}_t, \hat{R}_t)$ etc. Modifying the system yields

$$Z_t = \begin{pmatrix} x_t^e & x_t & x_{t-1} & x_{t-2} & \cdots & x_{t-(L-2)} \end{pmatrix}^T$$

$$Q_2 = \begin{pmatrix} (1-\omega)I_3 & \omega I_3 & 0 & \cdots & 0 & 0 \\ (1-\omega)M & \omega M + N_1 & N_2 & \cdots & N_{L-2} & N_{L-1} \\ 0 & I_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_3 & 0 \end{pmatrix}.$$

For stability, the roots of $P(\lambda) = Det[Q_2 - \lambda I_{3L}]$ must be inside the unit circle. One can show that

$$P(\lambda) = \lambda^{2L-2}(1-\omega-\lambda)\tilde{P}(\lambda)$$

Thus, the roots of $P(\lambda)$ are inside the unit circle if and only if the roots of $\tilde{P}(\lambda)$ are inside the unit circle. In the limit $\omega \to 0$, we have

$$\tilde{P}(\lambda) = (1-\lambda)^2(\lambda^{L-1} + h\mu \sum_{k=0}^{L-2} \mu^{L-2-k}\lambda^k)$$

where

$$h = \frac{\beta\kappa\psi_p y^*(y^*-\bar{g})}{\beta\kappa\psi_p y^*(y^*-\bar{g}) + (\beta\pi^*\psi_y(y^*-\bar{g}) + (\pi^*)^2\sigma y^*)\left(\sum_{i=0}^{L-1}\mu^i\right)} \in (0,1)$$

The polynomial has two unit roots. For the other roots one can apply a generalization of the classic Enerström-Kakeya theorem in Gardner and Govil (2014), Theorem 3.6, stating that the roots of the second polynomial in $\tilde{P}(\lambda)$ satisfy $|\lambda| < \mu < 1$.

Therefore, the roots of $P(\lambda)$ are inside the unit circle if $\partial\lambda/\partial\omega < 0$ evaluated at $\omega = 0$ and $\lambda = 1$. To evaluate the derivative, we consider the Taylor series expansion of $\tilde{P}(\lambda)$ up to second order at point (λ_0, ω_0). Let $(d\lambda, d\omega) = (\lambda, \omega) - (\lambda_0, \omega_0)$. Then

$$\begin{aligned}\tilde{P}(\lambda,\omega) &= \tilde{P}(\lambda_0,\omega_0) + \tilde{P}_\lambda(\lambda_0,\omega_0)d\lambda + \tilde{P}_\omega(\lambda_0,\omega_0)d\omega + \\ &\quad \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2} + Q,\end{aligned}$$

where subscripts denote partial derivatives and Q is a remainder.

Now

$$\begin{aligned}\tilde{P}_\omega(\lambda_0,\omega_0) &= 0 \\ \tilde{P}_\lambda(\lambda_0,\omega_0) &= 0\end{aligned}$$

so we get the approximation

$$\tilde{P}(\lambda,\omega) = \tilde{P}(\lambda_0,\omega_0) + \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2}.$$

Now impose

$$\tilde{P}(\lambda,\omega) - \tilde{P}(\lambda_0,\omega_0) = 0$$

to compute the derivative of the implicit function. So we have

$$\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0,\omega_0)\frac{d\omega^2}{2} + \tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)\frac{d\lambda^2}{2} = 0$$

or

$$\frac{d\lambda}{d\omega} = \frac{-1}{2}\left(\frac{\tilde{P}_{\omega\omega}(\lambda_0,\omega_0)}{\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)} + \frac{\tilde{P}_{\lambda\lambda}(\lambda_0,\omega_0)}{\tilde{P}_{\lambda\omega}(\lambda_0,\omega_0)}\left(\frac{d\lambda}{d\omega}\right)^2\right)$$

Evaluating the partial derivatives at $(\lambda_0, \omega_0) = (1, 0)$ we have

$$\begin{aligned}
\tilde{P}_{\omega\omega}(1,0) &= (-1)^L \frac{2(y^* - \bar{g})((1-\beta)\beta\pi^*\psi_y + \kappa y^*(\beta\psi_p - \pi^*))}{(\beta-1)^2\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g})/\left(\sum_{k=0}^{L-1}\mu^i\right) + (\pi^*)^2\sigma y^*\right)} \\
\tilde{P}_{\lambda\lambda}(1,0) &= (-1)^L \frac{2\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)}{\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g})/\left(\sum_{k=0}^{L-1}\mu^i\right) + (\pi^*)^2\sigma y^*} \\
\tilde{P}_{\lambda\omega}(1,0) &= (-1)^L \frac{\kappa y^*(y^*-\bar{g})(\pi^* - 2\beta\psi_p) + (2-\beta)\beta\pi^*\psi_y(\bar{g} - y^*) - (1-\beta)(\pi^*)^2\sigma y^*}{(\beta-1)\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g})/\left(\sum_{k=0}^{L-1}\mu^i\right) + (\pi^*)^2\sigma y^*\right)}
\end{aligned}$$

One can show that $\tilde{P}_{\omega\omega}(1,0) > 0$, $\tilde{P}_{\lambda\lambda}(1,0) > 0$, $\tilde{P}_{\lambda\omega}(1,0) > 0$ if L is even and $\tilde{P}_{\omega\omega}(1,0) < 0$, $\tilde{P}_{\lambda\lambda}(1,0) < 0$, $\tilde{P}_{\lambda\omega}(1,0) < 0$ if L is odd. Therefore, $\partial\lambda/\partial\omega < 0$ and we have stability for $\kappa \geq 0$ and small ω.

D.7 Proof of Proposition 8

In the linearization (31) we get[62]

$$DF_x = \begin{pmatrix} 1 & 0 & \frac{\beta(y^*-g)}{\pi^*\sigma} \\ -\kappa & 1 & 0 \\ -\frac{\psi_y}{y^*} & -\frac{w_c\psi_p}{\pi^*} & 1 \end{pmatrix}$$

$$DF_{x^e} = \begin{pmatrix} -1 & \frac{-(g-y^*)}{\pi^*\sigma(\beta-1)} & \frac{\beta^2(g-y^*)}{\pi^*\sigma(\beta-1)} \\ \frac{\beta\kappa}{\beta-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$DF_{cb} = \begin{pmatrix} 0 \\ 0 \\ -(1-w_c)\psi_p/\pi^* \end{pmatrix}$$

where

$$\kappa = \frac{\nu\left(\frac{(\nu-1)\sigma y^*(y^*-\bar{g})^{-\sigma-1}}{\nu} - \frac{(\nu-1)(y^*-\bar{g})^{-\sigma}}{\nu} + \frac{(\epsilon+1)y^{*\frac{\epsilon+1}{\alpha}-1}}{\alpha^2}\right)}{\gamma(2\pi^*-1)} \geq 0$$

[62]The proof applies to the slightly generalized case in which the interest rate rule can also respond to output.

if $\sigma > (y^* - \bar{g})/y^*$. It follows that

$$M = -(DF_x)^{-1} DF_{x^e} = \begin{pmatrix} \frac{y^*\left(w_c\beta^2\kappa\psi_p(y^*-\bar{g})+(\beta-1)\pi*^2\sigma\right)}{\eth_3} & \frac{\pi^* y^*(\bar{g}-y^*)}{\eth_3} & \frac{\beta^2\pi^* y^*(y^*-\bar{g})}{\eth_3} \\ \frac{\kappa(-\pi^*)\left(\beta^2\psi_y(y^*-\bar{g})+\pi^*\sigma y^*\right)}{\eth_3} & \frac{\kappa\pi^* y^*(\bar{g}-y^*)}{\eth_3} & \frac{\beta^2\kappa\pi^* y^*(y^*-\bar{g})}{\eth_3} \\ \frac{\pi^*\sigma((\beta-1)\pi^*\psi_y-w_c\kappa\psi_p y^*)}{\eth_3} & \frac{(\bar{g}-y^*)(\pi^*\psi_y+w_c\kappa\psi_p y^*)}{\eth_3} & \frac{\beta^2(y^*-\bar{g})(\pi^*\psi_y+w_c\kappa\psi_p y^*)}{\eth_3} \end{pmatrix},$$

$$N_{cb} = -(DF_x)^{-1} DF_{cb} = \begin{pmatrix} \frac{(w_c-1)\beta\psi_p y^*(\bar{g}-y^*)}{(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*)} \\ \frac{(w_c-1)\beta\kappa\psi_p y^*(\bar{g}-y^*)}{(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*)} \\ \frac{(w_c-1)\pi^*\sigma\psi_p y^*}{(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*)} \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & \frac{(w_c-1)\beta\psi_p y^*(\bar{g}-y^*)}{(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*)} & 0 \\ 0 & \frac{(w_c-1)\beta\kappa\psi_p y^*(\bar{g}-y^*)}{(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*)} & 0 \\ 0 & \frac{(w_c-1)\pi^*\sigma\psi_p y^*}{(\beta\pi^*\psi_y(y^*-\bar{g})+w_c\beta\kappa\psi_p y^*(y^*-\bar{g})+(\pi^*)^2\sigma y^*)} & 0 \end{pmatrix}$$

where $\eth_3 = (\beta - 1)\left(\beta\pi^*\psi_y(y^* - \bar{g}) + w_c\beta\kappa\psi_p y^*(y^* - \bar{g}) + (\pi^*)^2\sigma y^*\right) < 0$.

Introduce the notation $x_t = (\hat{y}_t, \hat{\pi}_t, \hat{R}_t)$ etc. Modifying the system (19), (31), and the linearization of (29) yields yields

$$Z_t = QZ_{t-1}, \text{ where} \tag{114}$$

$$Z_t = (\ x_t \quad x_t^e \quad \hat{\pi}_t^{cb}\ )^T$$

$$Q = \begin{pmatrix} \omega M + w_c N & (1-\omega)M & (1-w_c)N_{cb} \\ \omega I_3 & (1-\omega)I_3 & 0_{3\times 1} \\ 0 \ \ w_c \ \ 0 & \cdots & 1-w_c \end{pmatrix}.$$

For stability, the roots of $P(\lambda) = Det[Q - \lambda I_7]$ must be inside the unit circle. One can show that

$$P(\lambda) = \lambda^3(1 - \omega - \lambda)\tilde{P}(\lambda).$$

Thus, the roots of $P(\lambda)$ are inside the unit circle if and only if the roots of $\tilde{P}(\lambda)$ are inside the unit circle. In the limit $\omega \to 0$, we have

$$\tilde{P}(\lambda) = (1 - \lambda)^2(\lambda - \mu)$$

where $\mu = \frac{(1-w_c)\left(\beta\pi^*\psi_y(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)}{(y^*-\bar{g})(\beta\pi^*\psi_y+\beta\kappa w_c\psi_p y^*)+(\pi^*)^2\sigma y^*} < \frac{\left(\beta\pi^*\psi_y(y^*-\bar{g})+(\pi^*)^2\sigma y^*\right)}{(y^*-\bar{g})(\beta\pi^*\psi_y)+(\pi^*)^2\sigma y^*} = 1$. Therefore, the roots of $P(\lambda)$ are inside the unit circle if $\partial\lambda/\partial\omega < 0$ evaluated at $\omega = 0$ and $\lambda = 1$. To evaluate the derivative, we consider the Taylor series expansion of $\tilde{P}(\lambda)$ up to second order at point (λ_0, ω_0). Let $(d\lambda, d\omega) = (\lambda, \omega) - (\lambda_0, \omega_0)$. Then

$$\begin{aligned} \tilde{P}(\lambda, \omega) &= \tilde{P}(\lambda_0, \omega_0) + \tilde{P}_\lambda(\lambda_0, \omega_0)d\lambda + \tilde{P}_\omega(\lambda_0, \omega_0)d\omega + \\ &\quad \tilde{P}_{\lambda\lambda}(\lambda_0, \omega_0)\frac{d\lambda^2}{2} + \tilde{P}_{\lambda\omega}(\lambda_0, \omega_0)d\lambda d\omega + \tilde{P}_{\omega\omega}(\lambda_0, \omega_0)\frac{d\omega^2}{2} + Q, \end{aligned}$$

where subscripts denote partial derivatives and Q is a remainder.

Evaluating the partial derivatives at $(\lambda_0, \omega_0) = (1, 0)$ we have

$$\begin{aligned} \tilde{P}_\omega(1, 0) &= 0, \\ \tilde{P}_\lambda(1, 0) &= 0, \end{aligned}$$

and imposing

$$\tilde{P}(\lambda, \omega) - \tilde{P}(\lambda_0, \omega_0) = 0,$$

we get the approximation

$$\tilde{P}_{\lambda\omega}(1,0) d\lambda dw + \tilde{P}_{\omega\omega}(1,0)\frac{d\omega^2}{2} + \tilde{P}_{\lambda\lambda}(1,0)\frac{d\lambda^2}{2} = 0$$

or

$$\frac{d\lambda}{d\omega} = \frac{-1}{2}\left(\frac{\tilde{P}_{\omega\omega}(1,0)}{\tilde{P}_{\lambda\omega}(1,0)} + \frac{\tilde{P}_{\lambda\lambda}(1,0)}{\tilde{P}_{\lambda\omega}(1,0)}\left(\frac{d\lambda}{d\omega}\right)^2\right)$$

Further, we have

$$\begin{aligned}
\tilde{P}_{\omega\omega}(1,0) &= \frac{2w_c(y^* - \bar{g})((\beta-1)\beta\pi^*\psi_y + \kappa\pi^* y^* - \beta\kappa\psi_p y^*)}{(1-\beta)^{-1}\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa w_c\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)} \\
\tilde{P}_{\lambda\lambda}(1,0) &= -\frac{2w_c\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)}{\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa w_c\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)} \\
\tilde{P}_{\lambda\omega}(1,0) &= \frac{w_c y^*\left((\beta-1)(\pi^*)^2\sigma + (\beta-2)\beta\pi^*\psi_y + \kappa\pi^* y^* - 2\beta\kappa\psi_p y^*\right)}{(1-\beta)\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa w_c\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)} \\
&- \frac{\bar{g}w_c((\beta-2)\beta\pi^*\psi_y + \kappa\pi^* y^* - 2\beta\kappa\psi_p y^*)}{(1-\beta)\left(\beta\pi^*\psi_y(y^*-\bar{g}) + \beta\kappa w_c\psi_p y^*(y^*-\bar{g}) + (\pi^*)^2\sigma y^*\right)}
\end{aligned}$$

One can show that $\tilde{P}_{\omega\omega}(1,0) < 0$, $\tilde{P}_{\lambda\lambda}(1,0) < 0$, $\tilde{P}_{\lambda\omega}(1,0) < 0$ if $\beta\psi_p > \pi^*$. Therefore, $\partial\lambda/\partial\omega < 0$ and we have stability for small w and $\kappa > 0$.

In part (ii) with $\gamma = 0$ the dynamics of output expectations do not depend on the rest of the system and can be shown to be locally convergent. The linearization (31) becomes

$$\begin{aligned}
\tilde{M} &\equiv -(D\tilde{F}_x)^{-1}D\tilde{F}_{x^e} = \begin{pmatrix} \frac{\pi^*}{w_c\psi_p\beta(1-\beta)} & -\frac{\beta\pi^*}{w_c\psi_p(1-\beta)} \\ \frac{1}{\beta(1-\beta)} & -\frac{\beta}{(1-\beta)} \end{pmatrix} \text{ and} \\
\tilde{N} &\equiv -(D\tilde{F}_x)^{-1}D\tilde{F}_{x_i^\cdot} = \begin{pmatrix} \frac{w_c-1}{w_c} & 0 \\ 0 & 0 \end{pmatrix} \text{ and} \\
\tilde{N}_{cb} &\equiv -(D\tilde{F}_x)^{-1}D\tilde{F}_{x_i^\cdot} = \begin{pmatrix} \frac{w_c-1}{w_c} \\ 0 \end{pmatrix}, \; i = 1, ..., L-1.
\end{aligned}$$

Introduce the notation $\tilde{x}_t = (\hat{\pi}_t, \hat{R}_t)$ etc. Modifying the system (19), (31) and the linearization of (29) yields

$$\tilde{Z}_t = \tilde{Q}\tilde{Z}_{t-1}, \text{ where} \tag{115}$$

$$\tilde{Z}_t = \begin{pmatrix} x_t & x_t^e & \hat{\pi}_t^{cb} \end{pmatrix}^T$$

$$\tilde{Q} = \begin{pmatrix} \omega\tilde{M} + w_c\tilde{N} & (1-\omega)\tilde{M} & (1-w_c)\tilde{N}_{cb} \\ \omega I_2 & (1-\omega)I_2 & 0_{2\times 1} \\ w_c \;\; 0 & \cdots & 1-w_c \end{pmatrix}.$$

For stability, the roots of $P(\lambda) = Det[\tilde{Q} - \lambda I_5]$ must be inside the unit circle. One can show that

$$P(\lambda) = \lambda^2(1-\omega-\lambda)\tilde{P}(\lambda),$$

where

$$\tilde{P}(\lambda) = \lambda^2 + \frac{\beta w_c \psi_p(\beta+\omega-1) - \pi^*\omega}{(1-\beta)\beta w_c \psi_p}\lambda + \frac{\pi^*\omega(1-w_c)}{(1-\beta)\beta w_c \psi_p}.$$

Thus, the roots of $P(\lambda)$ are inside the unit circle if and only if the roots of $\tilde{P}(\lambda)$ are inside the unit circle.

Let $a_0 = \frac{\pi^*\omega(1-w_c)}{(1-\beta)\beta w_c\psi_p}$ and $a_1 = \frac{\beta w_c\psi_p(\beta+\omega-1)-\pi^*\omega}{(1-\beta)\beta w_c\psi_p}$. The roots of $\tilde{P}(\lambda)$ are inside the unit circle if and only if the Schur-Cohn condition, $|a_1| < 1+a_0 < 2$, is satisfied. The Schur-Cohn condition is satisfied if $\psi_p > \max[\frac{\pi^*(\omega/w_c)(1-w_c)}{(1-\beta)\beta}, \bar{R}]$ and $\omega < \frac{(1-\beta)\beta w_c\psi_p}{\beta w_c\psi_p - \pi^*}$ if $\psi_p > \pi^*/(\beta w_c)$ or $\omega > 0$ otherwise.

D.8 Proof of Proposition 9

Recall that agents' PLM for $z \in \{\hat{\pi}, \hat{R}\}$ is $z_t = a_z + b_z R_{t-1}$. This implies:

$$\begin{aligned}
\hat{R}^e_{t+j} &= \frac{1-b_R^{j+1}}{1-b_R}a_R + b_R^{j+1}\hat{R}_{t-1} \\
\hat{\pi}^e_{t+j} &= a_\pi + b_\pi \hat{R}^e_{t+j-1}
\end{aligned}$$

for $j \geq 0$. Substituting expectations into (81) and (82) yields the ALM:

$$\begin{aligned}
\hat{R}_t &= \frac{\beta^{-1}}{1-\beta}a_\pi + \left(\beta^{-1}b_\pi + \frac{\beta^{-1}b_\pi - 1}{1-b_R}\left(\frac{\beta}{1-\beta} - \frac{\beta b_R^2}{1-\beta b_R}\right)\right)a_R \\
&+ \frac{\beta^{-1}b_\pi - \beta b_R}{1-\beta b_R}b_R\hat{R}_{t-1} \\
\hat{\pi}_t &= \frac{\pi^*}{w_c\psi_p}\hat{R}_t + \pi^*\frac{w_c-1}{w_c\psi_p}\hat{R}_{t-1}
\end{aligned}$$

The T-map is given by:

$$\begin{aligned}
a_R &\to \frac{\beta^{-1}}{1-\beta}a_\pi + \left(\beta^{-1}b_\pi + \frac{\beta^{-1}b_\pi - 1}{1-b_R}\left(\frac{\beta}{1-\beta} - \frac{\beta b_R^2}{1-\beta b_R}\right)\right)a_R \\
a_\pi &\to \frac{\pi^*}{w_c\psi_p}\left(\frac{\beta^{-1}}{1-\beta}a_\pi + \left(\beta^{-1}b_\pi + \frac{\beta^{-1}b_\pi - 1}{1-b_R}\left(\frac{\beta}{1-\beta} - \frac{\beta b_R^2}{1-\beta b_R}\right)\right)a_R\right) \\
b_R &\to \frac{\beta^{-1}b_\pi - \beta b_R}{1-\beta b_R}b_R \\
b_\pi &\to \frac{\pi^*}{w_c\psi_p}\left(\frac{\beta^{-1}b_\pi - \beta b_R}{1-\beta b_R}b_R\right) + \pi^*\frac{w_c-1}{w_c\psi_p}
\end{aligned}$$

Computing the E-stability conditions for the unique bounded REE (which is characterized by $(a_\pi, a_R, b_\pi, b_R) = (0, 0, \frac{w_c-1}{w_c\psi_p}\pi^*, 0)$) from the T-map in the usual manner, Proposition 9 follows.

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Figures

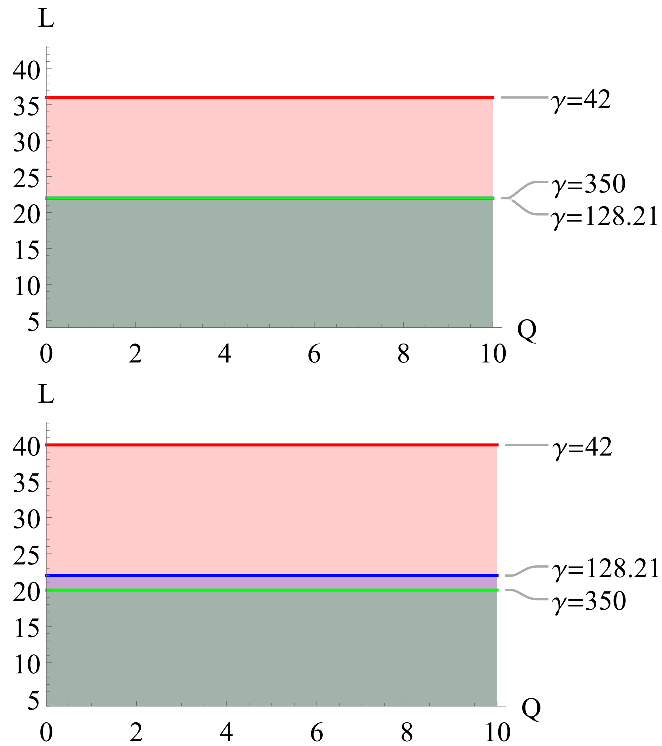

Figure I A,B: Stability of target steady state with overparameterized PLM

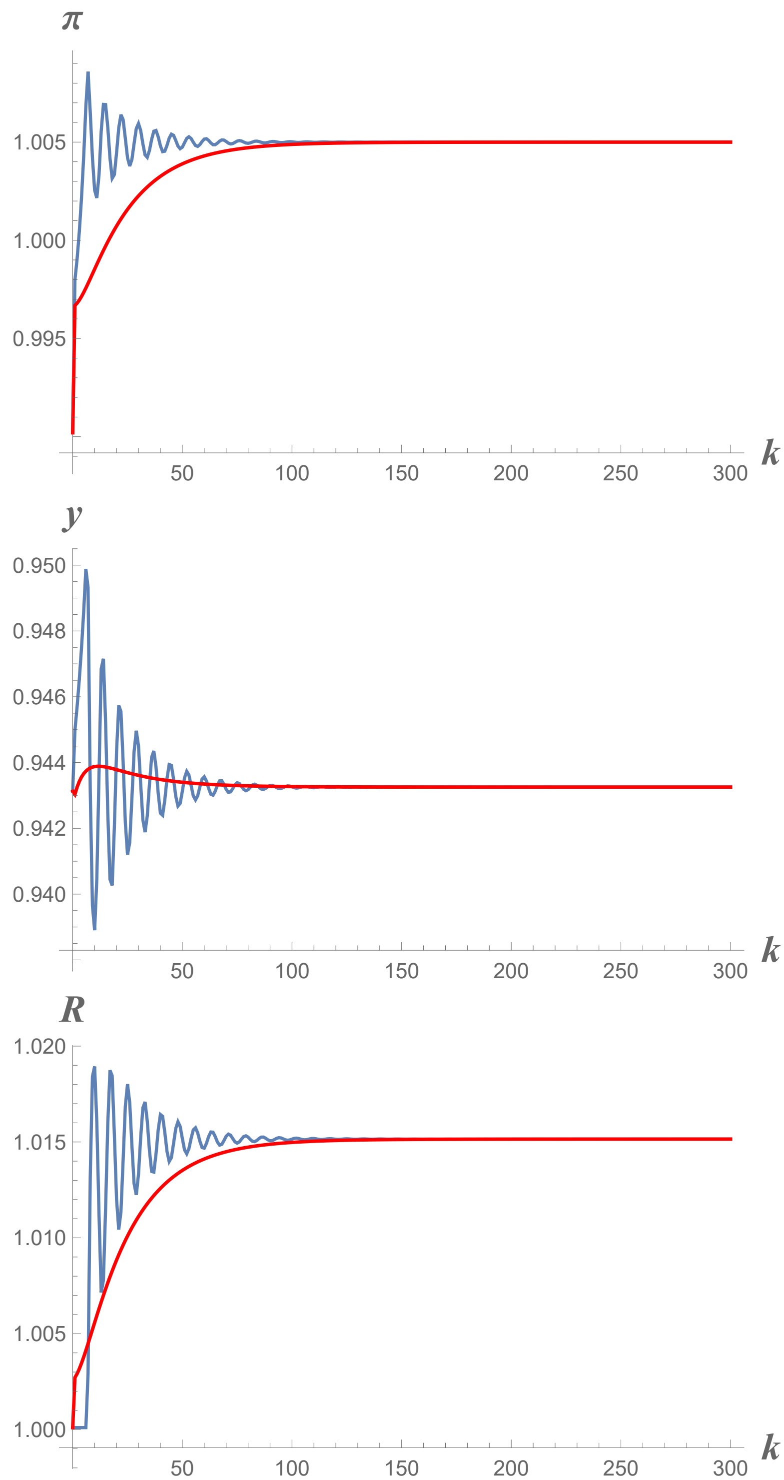

Figure II: Escape of inflation, output and interest rate from liquidity trap under AIT with transparency (blue) and IT (red)

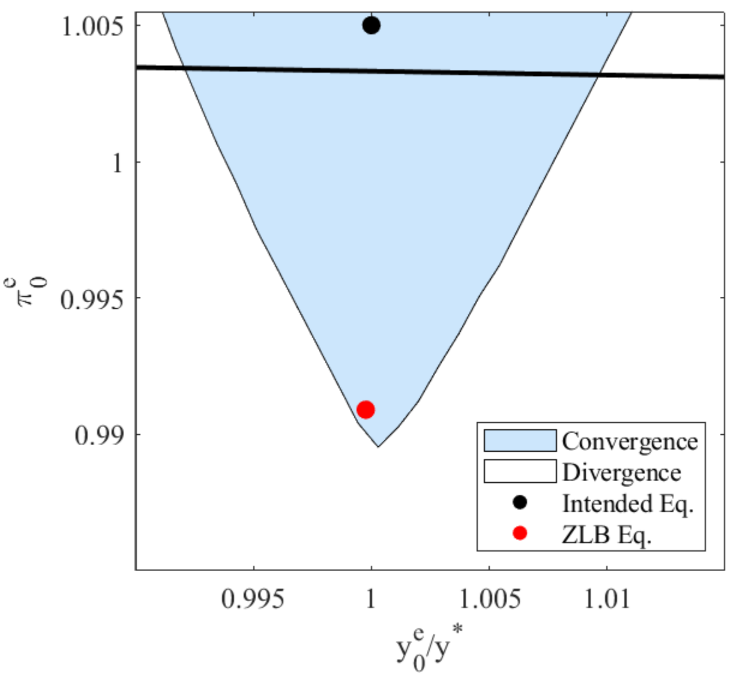

Figure III: Domain of escape to target steady state

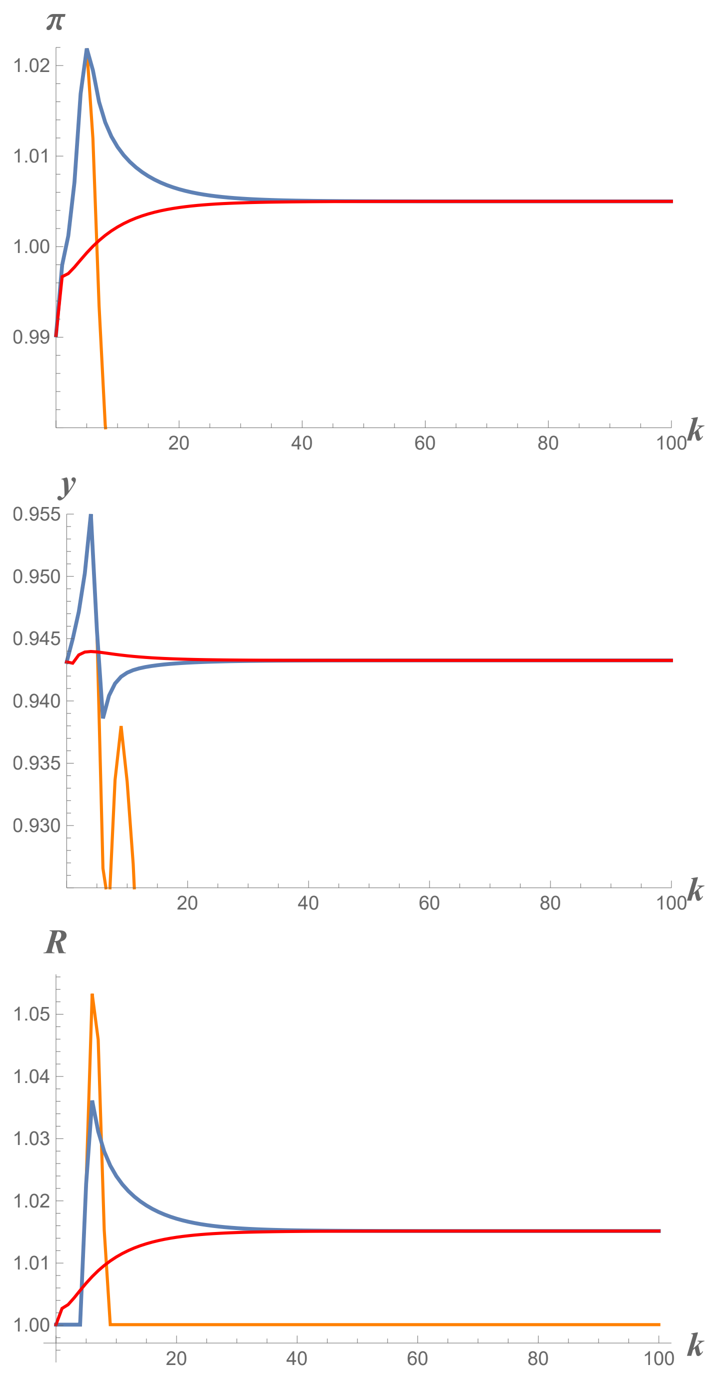

Figure IV: Escape of inflation, output and interest rate from liquidity trap under asymmetric AIT (blue), IT (red) and symmetric AIT (yellow)

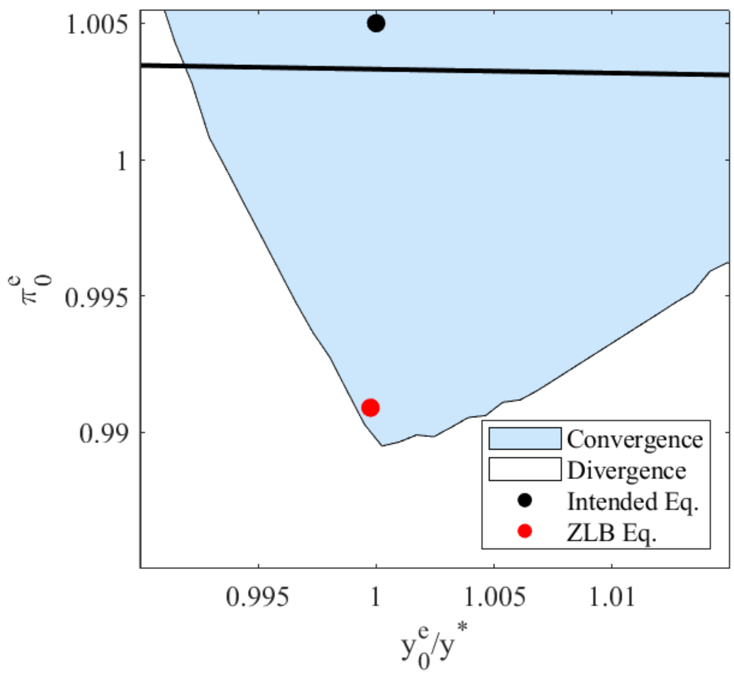

Figure A.1: Domain of escape for IT

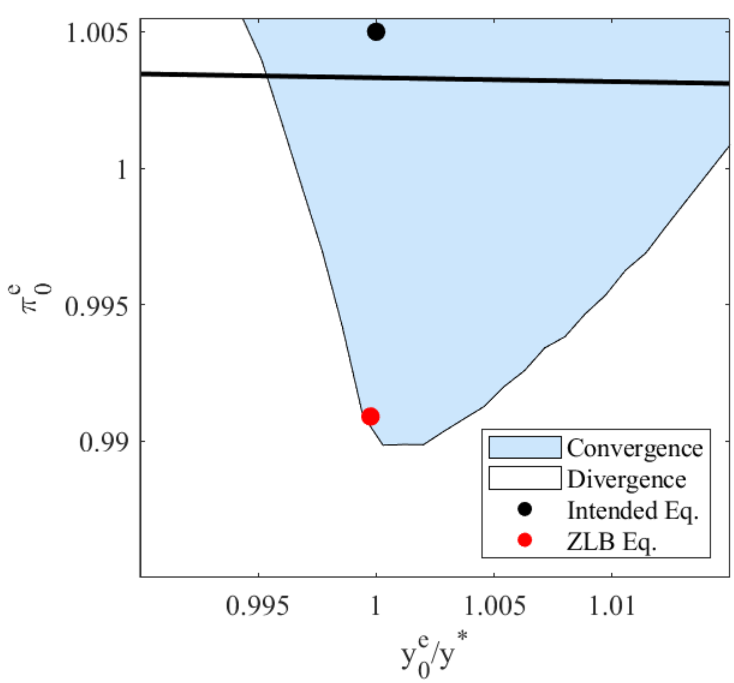

Figure A.2: Domain of escape for IT with MSV beliefs

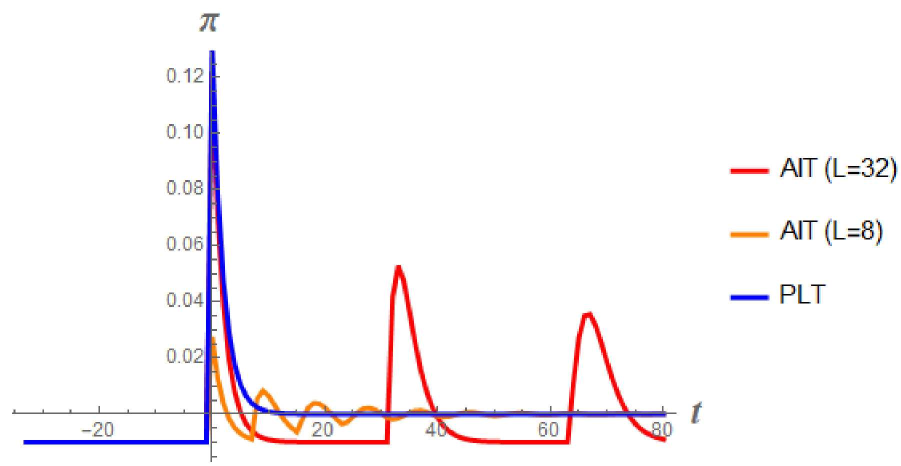

Figure A.3: Inflation under PLT (blue), AIT with $L = 8$ (orange) and AIT with $L = 32$ (red), sticky price model

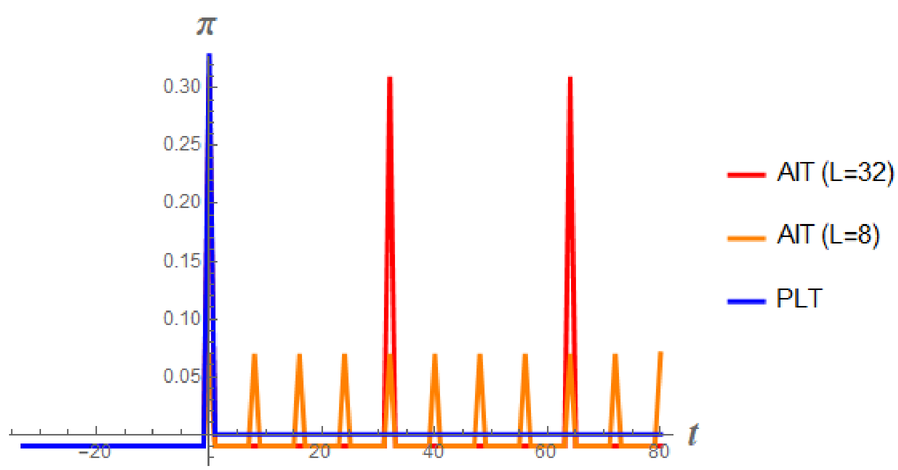

Figure A.4: Inflation under PLT (blue), AIT with $L = 8$ (orange) and AIT with $L = 32$ (red), flexible price model