E-stability vis-a-vis Determinacy in Markov-Switching DSGE Models*

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(Link to Most Recent Draft)

Abstract

This paper addresses the relationship between determinacy and E-stability in a general class of Markov-switching DSGE (MS-DSGE) models with lagged endogenous variables. We prove that determinacy conditions from Cho (2016) and Cho (2019) imply the E-stability of the unique solution when agents condition their expectations on contemporaneous variables and use one-step-ahead decision rules. We therefore extend the main result of McCallum (2007) to a class of DSGE models with time-varying parameters. As with linear DSGE models, E-stability conditions are weaker than determinacy conditions, but we show that MS-DSGE models present new cases where indeterminate models feature E-stable MSV solutions. In particular, we employ a New Keynesian model with recurring exogenous interest rate regimes and show that indeterminacy can obtain when our E-stability condition, which coincides with the Long Run Taylor Principle, is satisfied.

JEL Classification: C62, D83, D84, E42, E52

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1 Introduction

Rational expectations models admit multiple equilibria, and this forces researchers to confront the issue of equilibrium selection: which, if any, of a given model's rational expectations equilibria (REE) are economically reasonable? To that end, determinacy is used extensively as a selection criterion. When a model is determinate only one non-explosive REE exists, and this eliminates the need to choose between equilibria. Alternative criteria advance robustness to bounded rationality as an important equilibrium selection consideration. Notably, the adaptive learning literature advocates for the "learnability" or "E-stability" of a specific REE as a selection criterion. In this literature, rational agents are replaced by agents who do not know the economy's structure, and instead behave like econometricians who use standard tools, such as least squares, to learn the laws of motion for an economy's aggregate variables. As these agents learn, they use their perceived laws of motion to forecast aggregate variables in the economy, and they make decisions based on these forecasts. This process of learning using data generated by the learning process itself can destabilize agents' beliefs, but under reasonable assumptions,[1] agents beliefs will only converge to a REE if they converge at all. The necessary conditions for convergence to a REE are called the E-stability conditions.

Despite its popularity and importance, determinacy has some weaknesses as an equilibrium selection criterion. In particular, determinacy does not explain how agents coordinate on a unique REE. Instead, agents endowed with rational expectations are assumed to derive the equilibrium mappings in the economy from their sophisticated understanding of the economy's structure. Rational expectations is innocuous if real-life economic agents possess true beliefs about their economic environment, but the mere existence of disagreement among academic economists challenges the tenability of this assumption. If, instead, a REE is shown to be the outcome of a

[1] E.g. agents may be restricted to employ perceived laws of motion that possess the same functional form as the underlying REE.

learning process that resembles the process endured by real-life econometricians, then maybe we can rationalize how agents coordinate on said equilibrium. The E-stability conditions help to determine when a REE can be rationalized in such a way.

Because an E-stable determinate equilibrium is robust to some of determinacy's weaknesses, it is important to study and characterize the relationship between determinacy and E-stability. If we can isolate conditions under which determinacy and E-stability both obtain, then we can dispense with sometimes burdensome E-stability computations, and trust that our unique equilibrium is rationalizable as the outcome of a learning process. When the two criteria fail to select the same equilibrium, however, it should complicate our understanding of that equilibrium's reasonableness. For example, we might choose to reject an E-unstable determinate equilibrium on the grounds that nearly-rational agents would fail to produce the predicted REE dynamics. Additionally, we might give extra consideration to a uniquely E-stable equilibrium in an otherwise indeterminate model.

The relationship between determinacy and E-stability is extensively studied in linear rational expectations models.[2] McCallum (2007) shows that determinacy is sufficient for E-stability of the unique REE if agents observe contemporaneous variables and use one-step-ahead decision rules that depend on forecasts of next-period variable realizations.[3] Ellison and Pearlman (2011) identifies restrictions in agents' learning rules that extend McCallum (2007) to models with lagged information about endogenous variables. Bullard and Eusepi (2014) goes beyond Ellison and Pearlman (2011) and McCallum (2007) in at least two directions: (1) they allow for richer lag structures in the information set of agents; (2) they permit agents to form expectations over infinite-horizons. Their results are clear: determinacy does not generally

[2]Notable contributions include, but are not limited to, Evans and Honkapohja (2001), Bullard and Mitra (2002), McCallum (2007), McCallum (2009), Cochrane (2009), Ellison and Pearlman (2011), Bullard and Eusepi (2014).

[3]The most prominent example of a one-step-ahead decision rule is the household Euler equation. Alternatively, agents may utilize infinite-horizon decision rules, such as the consumption function. See Preston (2005) for more on infinite-horizon learning.

imply E-stability under infinite-horizon learning or finite-horizon learning. In particular, the presence of delays in the information set breaks the McCallum (2007) relationship between determinacy and E-stability in models of finite-horizon learning.

Determinacy is not well-understood in Markov-switching DSGE models, and consequently little has been said about the relationship between determinacy and E-stability in these environments. However, some recent research has made great strides in the direction of useful determinacy conditions.[4] Davig and Leeper (2007) provides the "Long Run Taylor Principle" (LRTP), which was initially considered a determinacy condition in the bounded stability sense, but was later shown to be insufficient by Farmer, Waggoner, and Zha (2010). Branch, Davig, and McGough (2013) shows that the LRTP of Davig and Leeper (2007) implies E-stability of the MSV solution under suitable assumptions in a class of *purely* forward-looking models. Farmer, Waggoner, and Zha (2009, 2011) spearhead the use of mean-square stability in the MS-DSGE literature by developing determinacy conditions in a class of forward-looking models and an implementable solution technique, but neither paper provides tractable determinacy conditions. Foerster, Rubio-Ramirez, Waggoner, and Zha (2016) use Gröbner bases to address determinacy within the class of minimal state variable solutions, but their methods are computationally costly and do not address sunspot indeterminacy. Barthelemy and Marx (2019) provides necessary and sufficient conditions for determinacy in the bounded stability sense. Cho (2016) develops both a tractable solution technique and sufficient conditions for determinacy in a general class of regime-switching DSGE models. Cho (2019) expands on the former paper to present necessary and sufficient conditions for determinacy in the mean-square stable sense.[5] Our work is most closely related to Cho (2016) and Cho (2019), though we

[4]Here we focus on the literature that explicitly studies determinacy in regime-switching DSGE models. We emphasize that many other works make important contributions to solving regime-switching DSGE models including Maih (2015) and Svensson and Williams (2007).

[5]Throughout this paper, we follow Cho (2016) and Cho (2019) and define a determinate model as a model that features a unique mean-square stable regime-dependent equilibrium. Intuitively, a

discuss connections between our findings and contributions in Barthelemy and Marx (2019) in section 5.

This paper explores the relationship between determinacy and E-stability in a very general class of Markov-switching rational expectations models with lagged endogenous variables. Specifically, we demonstrate that a set of tractable conditions for determinacy from Cho (2019) imply the learnability of the unique mean-square stable rational expectations solution if agents know current endogenous variables and use one-step-ahead rules such as Euler equations in their decision-making. This result contributes to our understanding of determinacy and E-stability in DSGE models in four ways. First, this result extends McCallum (2007), which finds that determinacy implies E-stability in a general class of linear rational expectations models, to environments with time-varying parameters. Second, our result extends Branch, Davig, and McGough (2013) to models with lagged endogenous variables.[6] Third, we argue that while E-stable MSV solutions to indeterminate linear DSGE models can exist, the conditions for E-stability in the MS-DSGE model class are considerably weaker than the conditions for determinacy. This allows us to identify new cases where E-stability holds and determinacy fails. We illustrate this using a model that features E-stable exogenous interest rate regimes despite model indeterminacy. In that model, the E-stability conditions coincide with the LRTP, as noted in Branch, Davig, and McGough (2013). Cho (2016) and Barthelemy and Marx (2019) also show that the LRTP can be satisfied in indeterminate models. Our contribution along these lines is to add economic meaning to those findings: the E-stability conditions for the MS-DSGE model class are substantially weaker than determinacy conditions. If we interpret these exogenous interest rate regimes as zero lower bound (ZLB) regimes, we furthermore arrive at a model of adaptive learning that features stable expecta-

stochastic process is mean-square stable if its first and second moments converge. We refer interested readers to Cho (2016) and Farmer, Waggoner, and Zha (2009) for more on mean-square stability in MS-DSGE models.

[6] Reed (2015) also studies determinacy and E-stability in regime-switching economies, but only examines purely forward-looking models.

tions at the ZLB. Finally, we provide evidence from real-time learning simulations that agents can learn the REE coefficients over time. In these simulations, beliefs converge to their RE values whenever E-stability conditions are satisfied and only when they are satisfied.

The paper is organized as follows. In section 2, we develop a general class of linear DSGE models, and discuss key results on determinacy and E-stability that are primarily developed by McCallum (2007). We then introduce the general class of Markov-switching DSGE models, some key results from the determinacy analysis of Cho (2016) and Cho (2019), and state our main result. Section 3 shows that while E-stability is weaker than determinacy in a class of linear DSGE models, it is much weaker than determinacy in our class of MS-DSGE models. For example, the LRTP, which is a special case of E-stability, is much weaker than determinacy. We illustrate cases where determinacy fails and E-stability holds. Section 4 presents numerical real-time learning simulations of learning agents in two DSGE models. Section 5 briefly discusses the potential link between the determinacy conditions in Barthelemy and Marx (2019) and E-stability. Section 6 concludes.

2 Determinacy and E-stability

This section analytically characterizes the determinacy and E-stability properties of a general class of models. First, we reproduce the main finding of McCallum (2007): a unique equilibrium is always E-stable when agents observe contemporaneous endogenous variables and use one-step-ahead forecasting rules. We then turn to a more general class of MS-DSGE models and show that determinacy conditions in Cho (2016) and Cho (2019) imply E-stability of the MSV solution under assumptions that are analogous to assumptions in McCallum (2007). This is the main contribution of this paper.

2.1 Linear DSGE Models

Research on the relationship between determinacy and E-stability often examines widely-used models of the following form:

$$x_t = ME_t x_{t+1} + Nx_{t-1} + Qu_t \tag{1}$$

where x_t is a $n \times 1$ vector of endogenous variables, u_t is a $m \times 1$ covariance-stationary process that follows:

$$u_t = \rho u_{t-1} + \epsilon_t$$

Following Cho (2016, 2019), we express a rational expectations solution to (1) as a linear combination of a minimal state variable (MSV) solution that depends on x_{t-1} and u_t and a non-fundamental solution component, w_t, such that:

$$x_t = \Omega x_{t-1} + \Gamma u_t + w_t \tag{2}$$

$$w_t = FE_t w_{t+1} \tag{3}$$

where the coefficient matrices satisfy the following conditions:

$$\begin{aligned} \Omega &= (I_n - M\Omega)^{-1} N \qquad (4) \\ \Gamma &= (I_n - M\Omega)^{-1} Q + F\Gamma\rho \\ F &= (I_n - M\Omega)^{-1} M \end{aligned}$$

We refer to (2) as the MSV solution when $w_t = 0$ for all t. Note that Γ and F are uniquely determined by Ω. Hence, we can index any MSV solution to (1) by the equilibrium coefficient matrix Ω. We can further characterize the set of solutions, $\mathcal{S}$, as follows:

$$\mathcal{S} = \{\Omega \in \mathcal{C}^{n\times n} | r(\Omega^1) \leq r(\Omega^2) \leq \ldots \leq r(\Omega^N)\} \tag{5}$$

where $r(A)$ denotes the spectral radius of A and N denotes the number of solutions to the fixed point problem (4). McCallum (2007) refers to Ω^1 as the minimum-of-modulus or MOD solution, and it is defined to be $\Omega^1 \in \mathcal{S}$ such that $r(\Omega^1)$ =min $r(\Omega)$ for all Ω in $\mathcal{S}$. In linear models of the form (1), we may employ a variety of techniques to identify the MOD solution, and its existence and uniqueness can be deduced from various properties of the model's eigenvalue-eigenvector system.[7] We explicitly refer to the MOD solution in this section because a unique equilibrium, when it exists, is always a MOD solution. As it turns out, we can decide whether a model (1) is determinate simply by identifying the MOD solution.

For the MOD solution to be the unique stable equilibrium, three things need to be true. First, $r(\Omega^1) < 1$ renders the solution non-explosive.[8] Second, the condition $r(F^1) < 1$ generates an explosive expectational difference equation for all non-fundamental solutions satisfying (3). Therefore this condition precludes coordination on sunspots of the form (3) in the MOD equilibrium. Finally, $1 \leq r(\Omega^2) \leq \ldots \leq r(\Omega^N)$ ensures that all MSV solutions, except for the MOD solution, are explosive. In principle, these conditions can be verified if one identifies $\mathcal{S}$. Such an exercise may prove costly and is inefficient relative to the familiar routines developed by Blanchard and Kahn (1980), Sims (2002) and other well-known works. McCallum (2007) and Cho (2019) provide the following succinct conditions for determinacy in linear models:

Theorem 1 *Consider the model (1) and suppose the MOD solution, Ω^1, exists and is real. (1) is a determinate model if and only if:*

1. $r(\Omega^1) < 1$

[7] For example, see Uhlig (1997), Klein (2000), Sims (2002).

[8] Here we use to "non-explosive" to mean "forward stable" or dynamically stable. We do not use "explosive" or "non-explosive" to refer to expectational stability.

2. $r(F^1) \leq 1$

Proof: See Proposition 3 in Cho (2019). ■

The determinacy conditions in Theorem 1 straightforwardly imply the E-stability of the unique MSV solution (MOD solution) given by Ω^1. To show this, we begin by replacing rational agents with learning agents who believe the economy evolves according to a *perceived* law of motion (PLM):

$$x_t = a + bx_{t-1} + cu_t \tag{6}$$

We assume that all agents in the economy have the same PLM and that agents observe all current variables when forming expectations (i.e. agents observe x_t and u_t when forming expectations at t). In what follows, we let $\hat{E}_t x_{t+1}$ denote the subjective expectations formed by the learning agents. Since these agents do not know the objective probability distributions for the model's variables and form subjective expectations using their PLM, we can express this expectations term as:

$$\hat{E}_t x_{t+1} = a + bx_t + c\rho u_t$$

It cannot be assumed *a priori* that agents' PLM coincides with the *actual* law of motion (ALM) that governs the equilibrium dynamics in the economy. The ALM is given by a version of (1) which replaces rational expectations with the aforementioned subjective expectations, yielding:

$$x_t = (I_n - Mb)^{-1} Ma + (I_n - Mb)^{-1} Nx_{t-1} + (I_n - Mb)^{-1} (Mc\rho + Q)u_t \tag{7}$$

The ALM implies a mapping from the set of beliefs $\Phi = (a, b, c)'$ to the actual equilibrium coefficients of the model, $T(\Phi)$. This mapping is referred to as the *T-map*. In our model, $T(\Phi) = ((I_n - Mb)^{-1} Ma, (I_n - Mb)^{-1} N, (I_n - Mb)^{-1} (Mc\rho + Q))'$.

If agents beliefs are consistent with (2), then the ALM becomes (2). That is, if $(a, b, c) = (0_{n\times 1}, \Omega, \Gamma) = \bar{\Phi}'$, then $T(\bar{\Phi}) = \bar{\Phi}$. This is another way of stating that rational agents possess true beliefs about the equilibrium mappings in the economy; rational agents believe in perceived laws of motions that are identical to the actual laws of motion. Of course even if our learning agents "learn" the REE coefficients, they will never truly be rational insofar as they will never learn the economy's structure. That is, the learnability of a REE does not by itself justify the behavioral primitives underlying rational expectations. If agents somehow learn the REE, however, it does suggest that there's something about the economy's structure which guides boundedly rational agents to the easy-to-model outcomes predicted by rational expectations analysis. To better understand those stabilizing aspects of an economy's structure we ask the question: when can agents learn to behave like rational agents? In other words, what moves (a, b, c) to $(0_{n\times 1}, \Omega, \Gamma)$ and under what conditions will such an evolution in beliefs occur? Evans and Honkapohja (2001) shows that a given REE is "E-stable", or, in other words, attainable as the limiting outcome of a real-time learning process if the E-stability conditions in Proposition 1 are satisfied.

Proposition 1 *Consider model (1), and assume that agents estimate a PLM of the form (6), observe all contemporaneous variables when forming expectations, and make decisions contingent on one-step-ahead decision rules (i.e. when the ALM is given by (7)). Then a REE,* $\bar{\Phi}' = (0_{n\times 1}, \Omega, \Gamma)$, *is said to be E-stable or stable under learning if the real parts of the following three matrices are less than one:*

1. $\Omega' \otimes F$
2. F
3. $\rho' \otimes F$

Proof: see Appendix A.1. ■

We are finally in a position to restate the main result from McCallum (2007) in Theorem 2.

Theorem 2 *Suppose agents estimate a PLM of the form (6), observe all contemporaneous variables when forming expectations, and make decisions contingent on one-step-ahead decision rules (i.e. the ALM is given by (7)). If (1) is determinate, then the unique equilibrium is E-stable.*

Proof: Determinacy requires $r(\Omega^1) < 1$ and $r(F^1) < 1$.[9] E-stability conditions 2 and 3 follow immediately from $r(F^1) < 1$ and $r(\rho) < 1$. Finally, $r(\Omega^{1'} \otimes F^1) = r(\Omega^1)r(F^1) < 1$. Hence, E-stability condition 1 follows from determinacy. ■

In sum, McCallum (2007) assesses E-stability and determinacy in models of the form (1) by stating both determinacy and E-stability conditions in terms of matrix functions of the MOD solution coefficients. Thus, the MOD solution concept offers a useful bridge between two of the most important equilibrium selection criteria in macroeconomics. In a class of MS-DSGE models that generalizes (1), Cho (2019) proposes a MOD solution concept, and a corresponding method that assesses determinacy in said model class using matrix functions of the MOD solution coefficients. Our contribution then establishes E-stability of the unique equilibrium of a MS-DSGE model by studying properties of the MOD solution. We show this in section 2.2.

2.2 Markov-switching DSGE Models and Main Proposition

In this section we introduce a general class of Markov-switching DSGE models. We then reproduce key results from Cho (2016) and Cho (2019) concerning the determinacy properties of these models. Finally, we extend McCallum (2007) by demonstrating that determinacy implies E-stability under the assumptions put forth in McCallum (2007). This is the main result of the paper.

[9] We abstract from the case $r(F^1) = 1$

In this paper, we study a general class of Markov-switching DSGE models that assume the form:

$$x_t = E_t(M(s_t, s_{t+1})x_{t+1}) + N(s_t)x_{t-1} + Q(s_t)u_t \tag{8}$$

where x_t is a $n \times 1$ vector of endogenous variables, u_t is a $m \times 1$ vector of exogenous variables that follows

$$u_t = \rho(s_t)u_{t-1} + \epsilon_t$$

where s_t is a S-state Markov Chain, $p_{ij} = Pr(s_{t+1} = j|s_t = i)$ is the (i, j)-th element of the transition probability matrix, P, $\rho(s_t)$ is a diagonal matrix and ϵ_t is a white noise process. By assumption, u_t is a mean-square stable process. In this paper, we use the mean-square stability concept, which is widely-used in the MS-DSGE literature. Intuitively, a n-dimensional discrete-time process, such as the RE solution to (10), is mean-square stable if its first and second moments are well-defined as t goes to infinity. More thorough descriptions of mean-square stability can be found in Costa, Fragoso, and Marques (2005), Farmer, Waggoner, and Zha (2009), and Cho (2016). From Theorem 1 in Cho (2016), any rational expectations solution to (8) can be written as a linear combination of a minimal state variable solution that depends on x_{t-1}, s_t, and u_t and a non-fundamental solution component, w_t, as:

$$x_t \;\; = \;\; \Omega(s_t)x_{t-1} + \Gamma(s_t)u_t + w_t \tag{9}$$

Cho (2016) provides a tractable method for obtaining solutions of the form (9), which we refer to as a MSV solution when $w_t = 0$. That paper also provides sufficient conditions for the existence of a unique mean-square stable MSV solution and non-existence of stable sunspot solutions of the form w_t. Cho (2019) shows that closely-related conditions, which we provide in Proposition 1, are not just sufficient but also necessary for ensuring the existence of a unique mean-square stable regime-dependent equilibrium (9). Cho (2019) furthermore generalizes the MOD solution concept to MS-DSGE models (i.e. the MOD solution is the most mean-square stable MSV solution). Cho (2019) emphasizes the MOD solution of (8), for similar reasons as section 2.1 emphasizes the MOD solution of (1): we can characterize a model's determinacy properties in terms of the properties of the MOD solution. Henceforth, we use $\Omega^1(s_t)$ for $s_t = 1, \dots, S$ to denote the MOD solution of (8). We refer interested readers to those papers, and Proposition 2 restates the determinacy conditions from Cho (2016, 2019).

Proposition 2 *Consider the model (8) and suppose the MOD solution, $\Omega^1(s_t)$, exists and is real-valued. (8) is a determinate model if and only if:*

1. $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) < 1$

2. $r(\Psi_{F^1 \otimes F^1}) \leq 1$

Proof: see Appendix C in Cho (2019).

Proposition 2 presents the Markov-switching DSGE analog of the determinacy conditions for the linear model presented in Theorem 1. We now derive the Markov-switching analog of the adaptive learning model developed in section 2.1. In our MS-DSGE setting, agents' PLM now assumes the form given by:

$$x_t = a(s_t) + b(s_t)x_{t-1} + c(s_t)u_t \tag{10}$$

Notice that agents use a PLM that shares the functional form of the MSV solution. We also assume that agents observe all contemporaneous variables including the Markov state, s_t, when forming expectations at time t and that agents are homogeneous. In this environment, subjective expectations, $\hat{E}_t x_{t+1}$, can be expressed as follows:

$$\begin{aligned}\hat{E}_t\left(M(s_t, s_{t+1})x_{t+1}\right) &= \hat{E}(M(s_t, s_{t+1})x_{t+1}|s_t = i; x_t, u_t) \\ &= \sum_{j=1}^{S} p_{ij} M(i,j)\left(a(j) + b(j)x_t + c(j)\rho(j)u_t\right)\end{aligned}$$

Substituting $\hat{E}_t(M(s_t, s_{t+1})x_{t+1})$ into (8) yields the ALM:

$$\begin{aligned}x_t &= \left(I - \sum_{j=1}^{S} p_{ij}M(i,j)b(j)\right)^{-1}\left(\sum_{j=1}^{S} p_{ij}M(i,j)a(j)\right) \\ &+ \left(I - \sum_{j=1}^{S} p_{ij}M(i,j)b(j)\right)^{-1} N(i)x_{t-1} \\ &+ \left(I - \sum_{j=1}^{S} p_{ij}M(i,j)b(j)\right)^{-1}\left(\sum_{j=1}^{S} p_{ij}M(i,j)c(j)\rho(j) + Q(i)\right)u_t \qquad (11)\end{aligned}$$

If we define $B = (b(1)\ b(2)\cdots b(S))$ and $\Xi(i,B) = \left(I - \sum_{j=1}^{S} p_{ij}M(i,j)b(j)\right)$ then the state-contingent T-map becomes:

$$\begin{aligned}a(i) &\to \Xi(i,B)^{-1}\sum_{j=1}^{S} p_{ij}M(i,j)a(j) \\ b(i) &\to \Xi(i,B)^{-1}N(i) \\ c(i) &\to \Xi(i,B)^{-1}\left(\sum_{j=1}^{S} p_{ij}M(i,j)c(j)\rho + Q(i)\right)\end{aligned}$$

What happens if agents learn to behave like rational agents in a REE, $\Omega(s_t)$ (such that $(a(s_t), b(s_t), c(s_t)) = (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$ for all s_t)? Cho (2016) shows that $\Omega(i) = \Xi(i,\Omega)^{-1}N(i)$ and $\Gamma(i) = \Xi(i,\Omega)^{-1}\left(\sum_{j=1}^{S} p_{ij}M(i,j)\Gamma(j)\rho + Q(i)\right)$ for $i = 1,\ldots,S$ where $\Omega = (\Omega(1),\ldots,\Omega(S))$. It immediately follows that $(a(s_t), b(s_t), c(s_t))$ $= (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$ is a fixed point of the T-map, just as a REE in the model class (1) is always a fixed point of the associated T-map. As in section 2.2., we must grapple with the stability of beliefs around these REE fixed points by asking: when will $(a(s_t), b(s_t), c(s_t)) \to (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$ for all s_t?

To answer this question we assume that agents estimate S *within-regime* linear systems. We specifically focus on algorithms of the form:

$$\Phi(s_t)_{ts} = \Phi(s_t)_{ts-1} + \psi_{ts} R(s_t)_{ts}^{-1} z_t (x_t - \Phi(s_t)'_{ts-1} z_t) \quad (12)$$

$$R(s_t)_{ts} = R(s_t)_{ts-1} + \psi_{ts}(z_t z_t' - R(s_t)_{ts-1}) \quad (13)$$

where $z_t = (1, x'_{t-1}, u'_t)'$ and ts is either the number of realizations of state s_t up until time t, or ts is simply equal to t.[10] (12) is a special case of the recursive *conditional* least squares (RCLS) algorithm developed in LeGland and Mevel (1997) for estimation in environments with hidden Markov states. Specifically, the RCLS converges to (12) if we set $R(s_t) = I$ for all s_t, define ψ_{st} appropriately,[11] and add s_t to agents' time-t information sets. It should also be noted that this algorithm is identical to the recursive least squares (RLS) algorithm, which is the workhorse estimation algorithm in the adaptive learning literature, when $S = 1$. We therefore view this recursive learning scheme as a natural extension of RLS to environments with Markov-switching parameters. That is, our approach attempts to be in keeping with the spirit of Evans and Honkapohja (2001). McClung (2019) and other ongoing work more fully explores the properties of these algorithms in models with hidden states.

Because agents are learning Φ via the recursives algorithms (12) and (13), it is clear that beliefs may only converge to REE (i.e. potential convergence points, $\bar{\Phi}$, must satisfy $\bar{\Phi} = T(\bar{\Phi})$ where T denotes the T-map). We therefore follow Branch, Davig, and McGough (2013) and apply the stochastic approximation approach in

[10] We define ts in this flexible manner in order to make the learning algorithm more general. Basic results in this paper do not depend on the definition of ts, provided that standard regularity assumptions concerning the asymptotic behavior of the gain parameter are satisfied.

[11] E.g., we let $\psi_{st} = t^{-\alpha}$ where $0 < \alpha \leq 1$

Evans and Honkapohja (2001) to our regime-switching environment. That is, we derive E-stability conditions from the T-map.

Proposition 3 *Consider model (8), and assume that agents estimate a PLM of the form (10), observe all contemporaneous variables when forming expectations, and make decisions contingent on one-step-ahead decision rules (i.e. when the ALM is given by (11)). Then a REE, $\bar{\Phi}(s_t)' = (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$ for all s_t, is said to be E-stable or stable under learning if the real parts of the following three matrices are less than one:*

1. $\Psi_{\Omega'\otimes F}$
2. Ψ_F
3. $\Psi_{\rho'\otimes F}$

where each of these matrices is defined in Appendix A.2.

Proof: see Appendix A.2. ■

These conditions are the Markov-switching DSGE analog to the conditions presented in McCallum (2007) and Proposition 1. Unsurprisingly, these conditions are identical to the aforementioned E-stability conditions for linear DSGE models when $S = 1$. Having derived the relevant E-stability conditions, we are now in a position to present Proposition 4, which is the main result.

Proposition 4 *Suppose agents estimate a PLM of the form (10), observe all contemporaneous variables when forming expectations, and make decisions contingent on one-step-ahead decision rules (i.e. the ALM is given by (11)). If (8) is determinate, then the unique equilibrium is E-stable.*

Proof: see Appendix A.3. ■

Figure 1 depicts the relationship between the determinacy properties of models (8) and the E-stability of the MSV solution under the assumptions stated in Proposition 4.

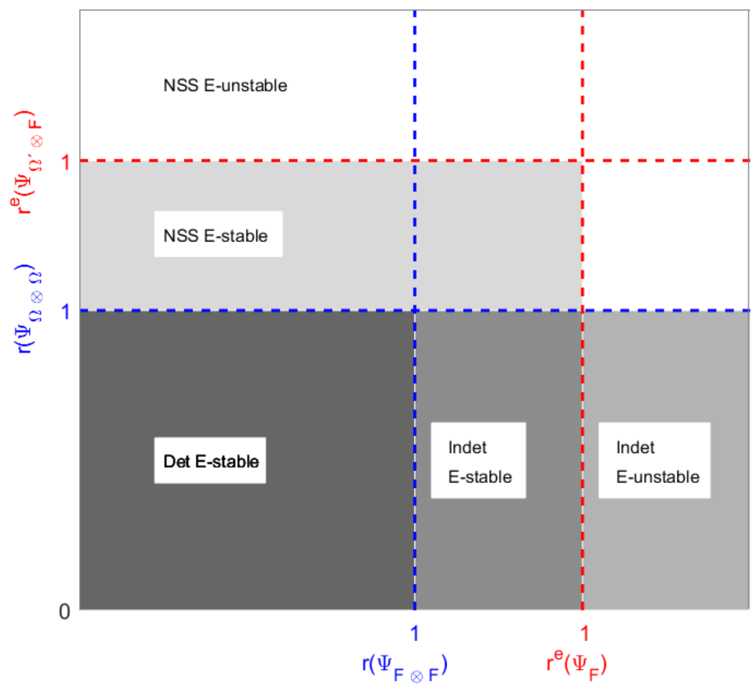

Figure 1: **Existence, Uniqueness and E-stability of Model Equilibria:** Here we suppose that $\Omega(s_t)$ denotes the MOD (minimum-of-modulus) solution (i.e. the most stable solution; see Cho (2019) for more). The MOD solution can be E-stable or E-unstable, and the underlying model is determinate (Det), indeterminate (Indet) or has no stable solutions (NSS). Solutions to the southwest of the red lines are E-stable, while solutions southwest of the blue lines are unique. Thus, determinacy is stronger than E-stability.

In linear DSGE models of the form (1), a variety of popular techniques are used to detect the existence and uniqueness of equilibrium.[12] Generally, it is more challenging to obtain MSV solutions in Markov-switching DSGE models, though popular techniques have been developed by Farmer, Waggoner, and Zha (2011), Maih (2015), Cho (2016), and Foerster, Rubio-Ramirez, Waggoner, and Zha (2016), among others.

[12]See, for example: Blanchard and Kahn (1980), Uhlig (1997), Sims (2002).

Applications in the remaining sections of this paper use the forward method in Cho (2016) to obtain MOD solutions. We find that the forward method is efficient, easy to implement, and effective in the class of models we consider.

3 Indeterminacy, E-stability and the LRTP

In this section, we argue that E-stability is much weaker than determinacy in the class of models given by (8). While E-stability is also weaker than determinacy in the class of linear DSGE models (1), Markov-switching parameters present new cases where E-stability holds and determinacy fails.

To illustrate this idea, we first consider model class (1). Recall that determinacy requires $r(\Omega) < 1$ and $r(F) < 1$, and E-stability requires $r^e(\Omega' \otimes F) < 1$ and $r^e(F) < 1$ where $r^e(A)$ denotes the maximum of the real parts of the eigenvalues of matrix A.[13] Since $r^e(A) \leq r(A)$, we can have indeterminacy and E-stability if $r^e(F) < 1 < r(F)$ and $r^e(\Omega \otimes F) < 1$ for some equilibrium given by (Ω, F), as McCallum (2007) argues.[14] Disagreement between the two equilibrium selection criteria even has policy implications in this setting. For example, Bullard and Mitra (2002) use a simple New Keynesian model to demonstrate a tendency for aggressive output-targeting to deliver E-stable, indeterminate models when central banks employ purely forward-looking Taylor rules. Such disagreements stem from the presence of negative feedbacks in the model dynamics, as captured by the condition $r^e(F) < -1$ (or the condition $r^e(\Omega) < -1$).

Now consider models of the form (8). From before, determinacy obtains if and only if $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ and $r(\Psi_{F\otimes F}) < 1$, and E-stability requires $r^e(\Psi_{\Omega'\otimes F}) < 1$ and $r^e(\Psi_F) < 1$. In the linear environment given by (1), $r(\Omega' \otimes F) = r(\Omega)r(F)$ and

[13] Since we assume stationarity of u_t in both model classes, we can dispense with the third E-stability matrices from Proposition 1 and Proposition 3 without affecting our results.

[14] Similarly, we can have an E-stable explosive solution when $1 < r(\Omega)$ and $r^e(\Omega \otimes F) < 1$.

$r(F \otimes F) < 1$ if and only if $r(F) < 1$. In the MS-DSGE environment modeled by (8), $r(\bar{\Psi}_{\Omega\otimes\Omega})r(\Psi_{F\otimes F}) > 1 > r(\Psi_{\Omega'\otimes F})$ and $r(\Psi_{F\otimes F}) > 1 > r(\Psi_F)$ are both possible. This suggests that we can more easily end up with E-stability and indeterminacy in MS-DSGE models without the negative feedbacks discussed above. For example, cases where $r(\bar{\Psi}_{\Omega\otimes\Omega}) > 1$ and $r(\bar{\Psi}_{F\otimes F}) > 1$, but $r^e(\Psi_{\Omega'\otimes F}) < 1$ cannot be ruled out *a priori*. Similarly, we can have indeterminacy with $r(\Psi_{F\otimes F}) > 1$, but E-stability can still hold if $r^e(\Psi_F) < 1$.

We can use a simple New Keynesian model to highlight cases where $0 \leq r^e(\Psi_F) < 1 < r^e(\Psi_{F\otimes F})$. Consider:

$$\begin{aligned}
y_t &= E_t y_{t+1} - \sigma^{-1}(i_t - E_t\pi_{t+1}) + r_t^n \\
\pi_t &= \beta E_t \pi_{t+1} + \kappa y_t + \mu_t \\
i_t &= \phi(s_t)\pi_t + \phi_y(s_t) y_t + \epsilon_t^m
\end{aligned}$$

where y is the output gap, π is inflation, i is the nominal interest rate, and r^n, μ, and ϵ^m are demand, supply, and monetary policy shocks, respectively. Under the fixed regime assumption that $\phi(s_t) = \phi \geq 0$ and $\phi_y(s_t) = \phi_y \geq 0$ for all s_t, Bullard and Mitra (2002) show that there is no region of the parameter space for which the model is indeterminate and admits an E-stable MSV solution. That is, the linear model is determinate if and only if it has an E-stable MSV solution if and only if $\phi > 1 - (1-\beta)\phi_y/\kappa$. Their result under this policy specification is extensively cited in support of the Taylor Principle as a guide to policy.

When we allow for switching in monetary policy parameters, we derive a nontrivial parameter space consistent with model indeterminacy and an E-stable MSV solution (see Figure 2).[15] As it turns out, the E-stability condition $r^e(\Psi_F) < 1$ is akin to the LRTP, which Cho (2016) shows to be $r(\Psi_F)$. Branch, Davig, and McGough (2013) assert a close relationship between E-stability of the MSV solution and the LRTP, while

[15] See Appendix A.4. for calibration details.

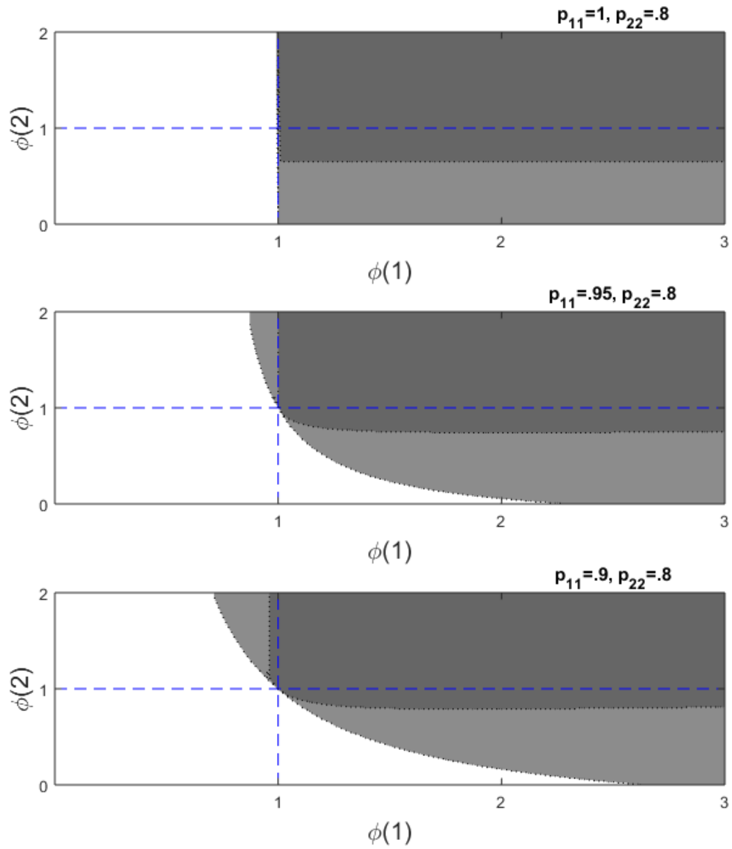

Figure 2: **Indeterminacy, E-stability and the LRTP:** The dark region is the determinacy region; the light gray region is the indeterminacy, E-stability region; white region is the E-instability, indeterminacy region. The dark region and light gray region show model parameterizations that satisfy the LRTP.

Cho (2016) and Barthelemy and Marx (2019) show that the LRTP is substantially weaker than the determinacy conditions developed in their respective papers. Our contribution in this section adds economic significance to their findings: the LRTP, which is a special case of E-stability, tells us when agents can coordinate on the MSV solution via adaptive learning mechanisms and, as it turns out, these conditions are substantially weaker than the conditions under which a unique equilibrium exists.

The weakness of E-stability conditions relative to determinacy may matter in settings that feature exogenous interest rate peg regimes (e.g. economies that feature recurring ZLB episodes). See, for example, panels A, B and C of Figure 2, which display E-stable, indeterminacy regions that intersect the $\phi(2) = 0$ axis (i.e. regime 2 features an exogenous interest rate peg or ZLB regime).[16] The quarterly model parameterization used in Figure 2 assumes an average ZLB regime duration of 5 quarters (i.e. $p_{22} = .8$), which reflects expectations that are arguably consistent with survey evidence presented in Swanson and Williams (2014).[17] Hence, our regime-switching model of adaptive learning helps to explain stability at the ZLB. Under rational expectations these ZLB regimes are subject to extrinsic sunspot volatility, and in most models of adaptive learning, the ZLB regime is also E-unstable.

What sets our model apart from standard linear models of adaptive learning is the fact that regime-switching learners explicitly account for the possibility of future regime change in their forecasts. This implies, for example, that agents could form expectations that the economy will revert to a monetary regime in which the Taylor Principle is satisfied (i.e. Regime 1), and these expectations of future active policy can help stabilize agents' current beliefs. On the other hand, linear least squares learners implicitly believe that the current structure of the economy is permanent– i.e. they have no prior beliefs about the nature of potential future changes.[18] It is this implicit belief that destabilizes agents' corresponding beliefs about the correlations in the model economy.

[16]To drive interest rates to the ZLB we could augment the model with a switching intercept term as in Bianchi and Melosi (2017) that pegs interest rates below steady state. It is straightforward to show that the corresponding E-stability conditions are unaffected by the inclusion of such an intercept term in the model.

[17]Swanson and Williams (2014) find that survey expectations of the ZLB duration in the U.S. fluctuated between 2 and 5 quarters prior to August 2011, when the Federal Reserve gave forward guidance that interest rates would remain at the ZLB through mid-2013.

[18]Even if agents implicitly account for the possibility of structural change by employing high gain parameters in their estimation routines, their forecasts assume that the current regime is in place forever.

4 Numerical Simulations

Many of the convergence theorems used in Evans and Honkapohja (2001) require the dynamics of our regressors, $z_t = (1, x'_{t-1}, u'_t)'$, to be linear conditional on beliefs in Φ. In models with Markov-switching parameters, fixing beliefs in (12) does not satisfy this assumption. While the main results in Evans and Honkapohja (2001) should hold under weaker assumptions (e.g. see Ljung (1975)), we nonetheless offer numerical support for our E-stability conditions in this section. We present such numerical support here for two models: (1) a simple univariate model; (2) a New Keynesian model with recurring changes in the monetary policy stance.

We find that agents' beliefs converge to the unique REE when determinacy (or, more generally, E-stability) obtains, and that beliefs always eventually diverge from any E-unstable REE. We also find that the rate of convergence of beliefs to their REE values is somewhat slower in our MS-DSGE simulations than what we might observe in some linear models. As in many linear models, the delayed convergence may have something to do with the presence of lagged endogenous variables, but it is also a consequence of two unique features of our regime-switching framework. First, our learning algorithm only updates beliefs corresponding to a given regime when that regime is active or "on-equilibrium" in the real-time learning simulation. For example, if $S = 2$ and $p_{11} = p_{22}$, then estimates corresponding to regime 1 are updated, on average, in half of the periods of a long simulation. In contrast, RLS algorithms will update a linear econometric model in every period. Second, agents' beliefs in the inactive or "off-equilibrium" regime always impact agents' forecasts, $\hat{E}_t x_{t+1}$, and therefore always affect the updating of estimates corresponding to the active regime. Hence, if agents' have non-rational beliefs in an inactive regime, it will delay the rate at which beliefs in the active regime converge to their REE values. This delayed rate of convergence may generate interesting and persistent learning dynamics in real-time simulations, and we leave this for future work.

4.1 Univariate Model

We now study the dynamics of learning agents' beliefs in a univariate model given by:

$$x_t = M(s_t)E_t x_{t+1} + N(s_t)x_{t-1} + \epsilon_t$$

where x is a scalar, and ϵ is a mean-zero *i.i.d.* disturbance to the economy. We solve the model using the forward method from Cho (2016) and Cho (2019), and we assume agents observe x_t contemporaneously, but we exclude ϵ from agents information sets in order to simplify our exposition. The inclusion of ϵ in agents' information sets, with or without appropriate modifications to agents' PLM, has no substantial effect on our results. Agents are assumed to use the following PLM:

$$x_t = a(s_t) + b(s_t)x_{t-1} + \epsilon_t^x$$

where ϵ_t^x is perceived *i.i.d* noise. We define $z_t = (1, x_{t-1})'$ and use the estimation algorithms (12) and (13). We give agents random initial beliefs in simulations, and use a decreasing gain parameter such that agents' beliefs are permitted to converge asymptotically. We also augment the learning algorithm with a projection facility that compels agents to reject explosive models of x and prevents beliefs from exploding outside of the so-called domain of attraction for beliefs around the REE.[19] Figure 3 shows a typical path for beliefs $(a(1), a(2), b(1), b(2))$ in a determinate model simulation in which agents are randomly endowed with incorrect (non-rational) initial beliefs (see Appendix A.4 for all model calibrations presented in Section 4). We know the MOD solution under study in Figure 3, $\Omega^1(s_t)$, is unique because $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) = 0.0462 < 1$, and $r(\Psi_{F^1 \otimes F^1}) = 0.4921 < 1$, and we know the equilibrium is E-stable because $r^e(\Psi_{\Omega^{1\prime} \otimes F^1}) < r^e(\Psi_{F^1}) = .6995 < 1$. Therefore, the conditions in Proposition 2 and Proposition 3 are satisfied.

[19]We very rarely invoke the projection facility in real-time learning simulations throughout this entire section.

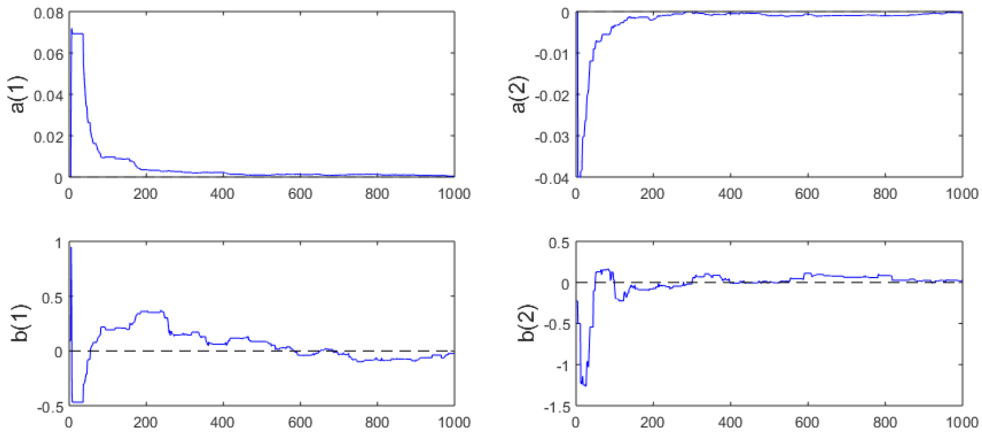

Figure 3: **Convergence to REE, Determinate Univariate Model:** deviations of agents beliefs, $(a(1), a(2), b(1), b(2))$ from their REE values are plotted here. Notice that beliefs are held fixed when they correspond to an inactive state (e.g. notice the flat "mesas" in the state 1 coefficients before t=50).

In Figure 3, the rate of convergence is somewhat slower than what we might observe in some linear models. This delayed convergence is not solely the result of regime-switching; univariate linear models with lagged endogenous variables can also feature delayed speed of convergence under RLS (e.g. see Figure 8.1. in Evans and Honkapohja (2001)). In fact, the multivariate model studied in section 4.2 features faster convergence of beliefs to their REE values despite the presence of regime changes. This notwithstanding, the rate of convergence of beliefs is slowed by the fact that off-equilibrium regime estimates are not updated by the algorithm each period (e.g. notice that the state 1 coefficient estimates are held fixed for a period before $t = 50$ when $s_t = 2$). Despite slow convergence, we nevertheless find strong numerical evidence that beliefs will eventually converge to their REE values in determinate univariate models.

In Figure 4, we explore learning dynamics in the vicinity of an E-unstable, indeterminate, and mean-square stable MOD solution to a univariate model. For this model, we know the MOD solution under study, $\Omega^1(s_t)$, is mean-square stable because $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) = .0463 < 1$, indeterminate because $r(\Psi_{F^1 \otimes F^1}) = 1.9272 > 1$, and

E-unstable because $r^e(\Psi_{F^1}) = 1.3286 > 1$. E-instability is evident in the drifting parameter estimates plotted in Figure 4. We see parameter estimates diverge from their true REE values in real-time learning simulations despite the fact that we give agents correct initial beliefs (i.e. we initialize beliefs at their REE values).

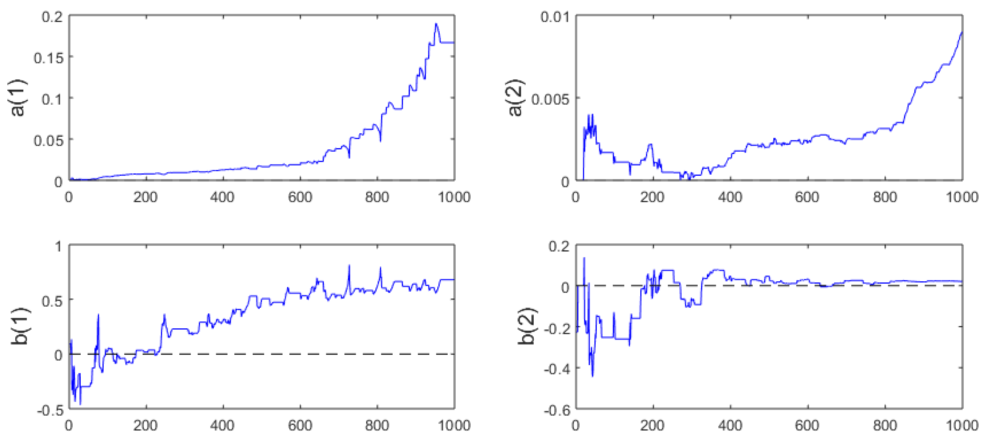

Figure 4: **Divergence from REE, Indeterminate Univariate Model:** deviations of agents beliefs, $(a(1), a(2), b(1), b(2))$ from their REE values are plotted here. Notice that beliefs are held fixed when they correspond to an inactive state (e.g. notice the flat "mesas" in the state 1 coefficients after t=950).

4.2 New Keynesian Model

We now demonstrate our results in a simple New Keynesian model:

$$
\begin{aligned}
y_t &= E_t y_{t+1} - \sigma^{-1}(i_t - E_t \pi_{t+1}) + r_t^n \\
\pi_t &= \beta E_t \pi_{t+1} + \kappa y_t + \mu_t \\
i_t &= \rho(s_t) i_{t-1} + (1 - \rho(s_t))\phi(s_t)\pi_t + \epsilon_t^m
\end{aligned}
$$

The economy in this model is subject to recurring monetary policy regime changes, which follow a 2-state Markov process, s_t. If $s_t = 1$ then monetary policy is active $(\phi(1) > 1)$. Monetary policy is passive $(\phi(2) < 1)$ if $s_t = 2$.

Like the model studied in section 4.1, this model can be written in the form (8). We solve this model using the forward method developed by Cho (2016) and Cho (2019). In learning simulations, agents' PLM is given by econometric model (10), and we assume that agents observe all contemporaneous model variables and update parameters according to (12) and (13). Furthermore, we initialize beliefs randomly in order to generate a meaningful learning problem for our boundedly rational agents. As before, we augment our learning model with a projection facility but find that the projection facility is used less than .1% of the time in simulations.

Figure 5 presents a typical time path for beliefs in this model when the underlying equilibrium, $\Omega^1(s_t)$, is unique and therefore E-stable. Again, calibration details for all parameterizations considered in this section are presented in Appendix A.4.[20] For this model, we know the equilibrium is unique because $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) = .4423 < 1$, and $r(\Psi_{F^1 \otimes F^1}) = .8795 < 1$, and the equilibrium is E-stable because $r^e(\Psi_{\Omega^{1\prime} \otimes F^1}) < r^e(\Psi_{F^1}) = .9326 < 1$. While we initialize beliefs away from the true REE coefficients for the model, we see that agents' beliefs under learning nonetheless converge to their REE values.

In section 3, we argue that the conditions for E-stability are substantially weaker than conditions for determinacy. Figure 6 presents a typical time path for beliefs in this model when the underlying MOD equilibrium, $\Omega^1(s_t)$, is E-stable but the model is indeterminate. Calibration details are, again, presented in Appendix A.4. For this model, we know the MOD solution is mean-square stable but not unique because $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) = .4422 < 1$, and $r(\Psi_{F^1 \otimes F^1}) = 1.0205 > 1$, and E-stable because $r^e(\Psi_{\Omega^{1\prime} \otimes F^1}) < r^e(\Psi_{F^1}) = .9974 < 1$.

Finally, Figure 7 displays time paths for beliefs in real-time learning simulations for three separate model parameterizations. The first parameterization is the parameter-

[20]We set $p_{11} = p_{22} = .95$ in the New Keynesian Model and set $p_{11} = p_{22} = .9$ in the univariate model, and this explains why off-equilibrium regime estimates are updated less frequently in Figures 5 through 7 than in Figures 3 and 4. Our results are robust to variation in transition probabilities.

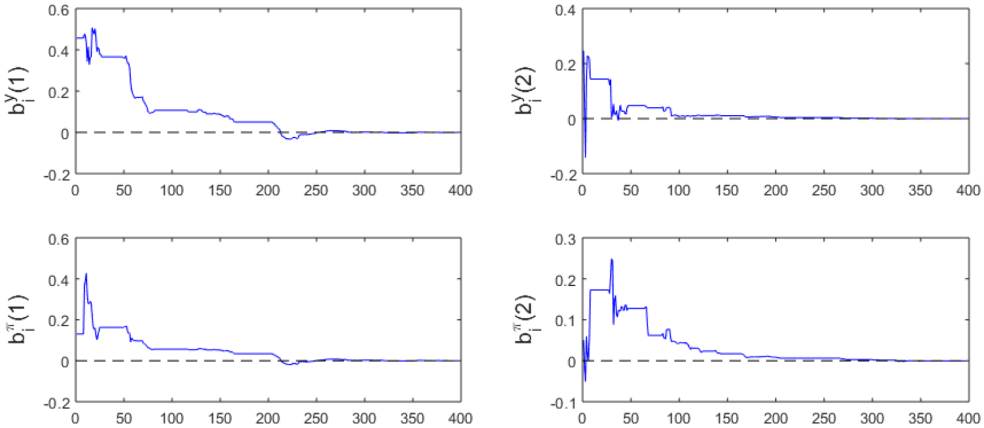

Figure 5: **Convergence to REE, New Keynesian Model:** deviations of agents beliefs from their REE values are plotted here. $b_i^y(s_t)$ ($b_i^\pi(s_t)$) are deviations of agents beliefs about output (inflation) dependence on lagged interest rates from REE values. Notice that beliefs are held fixed when they correspond to an inactive state (e.g. notice the flat "mesas" in inactive regimes)

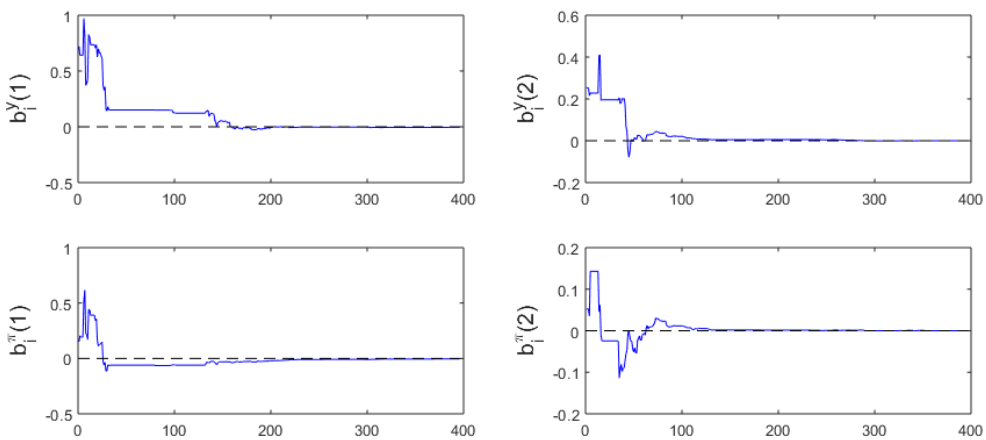

Figure 6: **Convergence to REE, New Keynesian Model:** deviations of agents beliefs from their REE values are plotted here. $b_i^y(s_t)$ ($b_i^\pi(s_t)$) are deviations of agents beliefs about output (inflation) dependence on lagged interest rates from REE values. Notice that beliefs are held fixed when they correspond to an inactive state (e.g. notice the flat "mesas" in inactive regimes)

ization in Figure 6 (i.e. the first parameterization yields an indeterminate, E-stable MOD solution). The second parameterization is identical to the paramaterization from Figure 6 except for one parameter: in Figure 6, the parameterization assumes $\phi(2) = .825$, whereas the second parameterization of Figure 7 assumes $\phi(2) = .8$. This small change in the second parameterization renders the MOD solution of the underlying model both E-unstable and indeterminate (i.e. $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) = .4422 < 1$, $r(\Psi_{F^1 \otimes F^1}) = 1.0364 > 1$, and $r^e(\Psi_{F^1}) = 1.005 > 1$). The third parameterization sets $\phi(2) = 0$, which makes the underlying MOD solution more E-unstable (i.e. $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) = .4411 < 1$, $r(\Psi_{F^1 \otimes F^1}) = 1.4781 > 1$, and $r^e(\Psi_{F^1}) = 1.1937 > 1$).

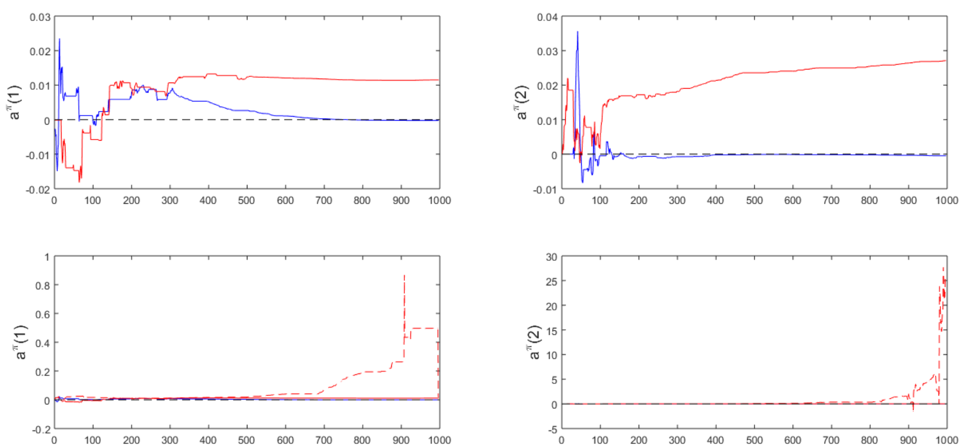

Figure 7: **Convergence to and Divergence from REE, New Keynesian Model:** deviations of agents beliefs from their REE values are plotted here. The blue line corresponds to the E-stable, indeterminate model parameterization from Figure 6. The solid red line corresponds to the the second model parameterization that features barely passive monetary policy and therefore delivers a MOD solution that barely violates the E-stability criterion. The dashed red line corresponds to the third parameterization that features a very passive monetary policy regime and therefore strongly violates the E-stability criterion. In each of the three simulations we draw different Markov state histories and different shocks.

The top two panels of Figure 7 compare time paths for beliefs for the first two parameterizations. Unsurprisingly, beliefs are diverging from the underlying REE in the second parameterization and converging to the REE in the first equilibrium.

Recall that the difference between these two parameterizations is very small, yet this change is sufficient to render the model E-unstable and therefore beliefs drift slowly away from their REE values. The bottom two panels reveal that beliefs diverge more dramatically from their REE values in the third parameterization, which features stronger passive monetary policy in the passive monetary regime. These last two panels therefore show that the magnitude of the eigenvalues associated to the model by the E-stability criterion can help predict the rate at which beliefs explode from an underlying REE.

5 Barthelemy and Marx (2019)

Barthelemy and Marx (2019) provide necessary and sufficient conditions for determinacy in the bounded stability sense for a related class of MS-DSGE models. While our main contribution is to argue that the unique *mean-square stable* solution selected by Cho (2016) and Cho (2019) is E-stable,[21] we nonetheless want to consider the potential E-stability of the unique bounded solution. Barthelemy and Marx (2019) confirms that a unique equilibrium assumes the form (9) (with $w_t = 0$) when it exists, and this means that our E-stability conditions apply. However, their approach also involves the computation of a limit which depends on an infinite number of different Markov state histories, and this fact makes it difficult to analytically characterize the relationship between their determinacy conditions and the E-stability conditions presented in this paper.

Because we have not proved that determinacy in the bounded stability sense implies E-stability, an interesting question remains: can a unique equilibrium in the bounded stability sense (i.e. a MSV solution selected by the Barthelemy and

[21] Cho (2016), Cho (2019), Farmer, Waggoner, and Zha (2009) provide conditions for determinacy in the mean-square stable sense. See Farmer, Waggoner, and Zha (2009) for more on the differences between mean-square stability and bounded stability in MS-DSGE models.

Marx (2019) criterion) be E-unstable under our assumptions about information and decision rules? If such an equilibrium, $\Omega^*(s_t)$, exists, then the following is true: $r^e(\Psi_{\Omega^{*\prime}\otimes F^*}) > 1$ or $r^e(\Psi_{F^*}) > 1$. If $r^e(\Psi_{\Omega^{*\prime}\otimes F}) > 1$ or $r^e(\Psi_{F^*}) > 1$ then $r(\Psi_{\Omega^*\otimes\Omega^*}) > 1$ or $r(\Psi_{F^*\otimes F^*}) > 1$. Hence, an E-unstable unique bounded equilibrium is (a) not mean-square stable; or (b) permits mean-square stable sunspot solutions. Regarding (a), a mean-square stable solution has a well-defined first and second moments, and a mean-square unstable solution does not. In practice, we should expect bounded processes to have well-defined first and second moments, but should we encounter a unique bounded equilibrium that is not mean-square stable, we might consider it a poorly-behaved solution due to its inability to deliver well-defined first and second moments for the model's variables. Similarly regarding (b) we might be worried about sunspot indeterminacy if non-fundamental solutions with well-defined first and second moments exist. Apart from these cases, which may give us reasons to further scrutinize the underlying equilibrium, and which we have not found in practice, the main message of this paper holds: when a MS-DSGE model admits a unique equilibrium, it is stable under learning.

6 Conclusion

Rational expectations models have multiple solutions, and this forces researchers to confront the issue of equilibrium selection. Most macroeconomic research uses two selection criteria: (1) determinacy; (2) E-stability. Determinacy is the most popular criterion, but the criterion itself cannot explain how agents coordinate on a unique REE. In contrast, E-stability selects equilibria that emerge as the outcome of an econometric learning process involving boundedly rational agents. Because an E-stable determinate equilibrium is robust to some of determinacy's weaknesses, it is important to study and characterize the relationship between determinacy and E-stability. If we can isolate conditions under which determinacy and E-stability both obtain, then we can dispense with sometimes burdensome E-stability computations, and trust that our unique equilibrium is rationalizable as the outcome of a learning process. When the two criteria fail to select the same equilibrium, however, it should complicate our understanding of that equilibrium's reasonableness.

In this paper we study determinacy and E-stability in a very general class of Markov-switching rational expectations models with lagged endogenous variables. Specifically, we demonstrate that a set of tractable conditions for determinacy from Cho (2019) imply the learnability of the unique mean-square stable rational expectations solution if agents know current endogenous variables and use one-step-ahead rules such as Euler equations in their decision-making. Our main contribution extends McCallum (2007), which finds that determinacy implies E-stability in a general class of linear rational expectations models, to environments with time-varying parameters. Additionally, this result extends Branch, Davig, and McGough (2013) to models with lagged endogenous variables.

We also argue that while E-stable MSV solutions to indeterminate linear DSGE models can exist, the conditions for E-stability in the MS-DSGE model class are considerably weaker than the conditions for determinacy. This allows us to identify new cases where E-stability holds and determinacy fails. We illustrate this using a model that features E-stable exogenous interest rate regimes despite model indeterminacy. In that model, the E-stability conditions coincide with the LRTP. If we interpret these exogenous interest rate regimes as ZLB regimes, we furthermore arrive at a model that features stable expectations at the ZLB. Finally, we provide evidence from real-time learning simulations that agents can learn the REE coefficients over time. In these simulations, beliefs converge to their RE values whenever E-stability conditions are satisfied and only when they are satisfied.

Appendix

A.1. Proof of Proposition 1

In this section we derive the E-stability conditions stated in Proposition 1. More specifically, we derive matrices 1-3 in Proposition 1, and a straightforward application of the E-stability Principle completes the proof. Please note that this proof is also demonstrated in Evans and Honkapohja (2001), p. 238.

We define $\Xi(b) = (I - Mb)$. The state-contingent T-map is given by:

$$\begin{aligned} a &\rightarrow \Xi(b)^{-1} Ma \\ b &\rightarrow \Xi(b)^{-1} N \\ c &\rightarrow \Xi(b)^{-1} (Mc\rho + Q) \end{aligned}$$

We can express the T-map as $T(a, b, c) = (T_a(a, b), T_b(b), T_c(b, c))$. Define $\Phi = (a, b, c)$, and denote the REE of interest $\bar{\Phi} = (\bar{a}, \bar{b}, \bar{c}) = (0_{n\times 1}, \Omega, \Gamma)$. Our task is to compute $DT(\bar{\Phi})$ where $DT(\Phi) = \partial T/\partial\Phi$. By the E-stability Principle, $\bar{\Phi}$ is E-stable if the real parts of the eigenvalues of the matrices comprising $DT(\bar{\Phi})$ are less than one.

Again, it is helpful to consider $T(a, b, c) = (T_a(a, b), T_b(b), T_c(b, c))$. We compute $DT(\bar{\Phi})$ in three stages:

1. Since the system $T_b(b)$ decouples from the rest of the T-map equations, we compute $DT_b(\bar{b})$ where $DT_b(b) = \partial T_b/\partial b$ and establish conditions under which $b \rightarrow \bar{b} = T_b(\bar{b})$.

2. Having established stability of beliefs b under learning, we compute $DT_a(\bar{a}, \bar{b})$ where $DT_a(a, b) = \partial T_a/\partial a$ and determine when $a \rightarrow \bar{a} = T_a(\bar{a}, \bar{b})$.

3. Having established stability of beliefs b under learning, we compute $DT_c(\bar{b}, \bar{c})$

where $DT_c(b,c) = \partial T_c/\partial c$ and determine when $c \rightarrow \bar{c} = T_c(\bar{b},\bar{c})$.

To solve for $DT_b(\bar{b})$, we linearize $T_b(b)$ at the REE and vectorize the resulting equation. We then use the following identification rule: if $vec(dT_b) = Avec(db)$ then $A = DT_b(b)$ where dT_b is the linearized system of equations. We obtain:

$$DT_b(\bar{b}) \quad = \quad \Omega' \otimes F$$

E-stability requires the real parts of $\Omega' \otimes F$ to be less than one. We now turn to the equation for a:

$$T_a(a,b) = \Xi(b)^{-1} Ma$$

Straightforward matrix calculus yields:

$$DT_a(\bar{a},\bar{b}) = F$$

E-stability requires the real parts of F to be less than one. Finally, we consider the equation for c:

$$T_c(b,c) \quad = \quad \Xi(b)^{-1}(Mc\rho + Q)$$

Using the same methods as before we obtain the following Jacobian evaluated at the REE where $\bar{c} = \Gamma$:

$$DT_c(\bar{b},\bar{c}) = \rho' \otimes F$$

E-stability therefore requires the real parts of $\rho \otimes F$ to be less than one. ■

A.2. Proof of Proposition 3

In this section we derive the E-stability conditions stated in Proposition 3. More specifically, we derive matrices 1-3 in Proposition 3, and a straightforward application of the E-stability Principle completes the proof.

We define $B = (b(1)\ b(2) \cdots b(S))$ and $\Xi(i, B)$ as in section 2.2, and let 0_n denote $n \times n$ matrix of zeros. Additionally, define $A = (a(1)'\ a(2)' \cdots a(S)')'$ and $C = (c(1)\ c(2) \cdots c(S))$. The state-contingent T-map is given by:

$$
\begin{aligned}
a(i) &\rightarrow \Xi(i, B)^{-1} \sum_{j=1}^{S} p_{ij} M(i, j) a(j) \\
b(i) &\rightarrow \Xi(i, B)^{-1} N(i) \\
c(i) &\rightarrow \Xi(i, B)^{-1} \left(\sum_{j=1}^{S} p_{ij} M(i, j) c(j) \rho(j) + Q(i) \right)
\end{aligned}
$$

We can express the T-map as $T(A, B, C) = (T_A(A, B), T_B(B), T_C(B, C))$. Define $\Phi = (A, B, C)$, and denote the REE of interest $\bar{\Phi} = (\bar{A}, \bar{B}, \bar{C}) = (0_{nS\times 1}, \Omega, \Gamma)$ where the matrices Ω and Γ collect the state-dependent rational expectations coefficients $\Omega(s_t)$ and $\Gamma(s_t)$, and are conformable to B and C respectively. Our task is to compute $DT(\bar{\Phi})$ where $DT(\Phi) = \partial T / \partial \Phi$. By the E-stability Principle, $\bar{\Phi}$ is E-stable if the real parts of the eigenvalues of the matrices comprising $DT(\bar{\Phi})$ are less than one.

Again, it is helpful to consider $T(A, B, C) = (T_A(A, B), T_B(B), T_C(B, C))$. We compute $DT(\bar{\Phi})$ in three stages:

1. Since the system $T_B(B)$ decouples from the rest of the T-map equations, we compute $DT_B(\bar{B})$ where $DT_B(B) = \partial T_B / \partial B$ and establish conditions under which $B \rightarrow \bar{B} = T_B(\bar{B})$.

2. Having established stability of beliefs B under learning, we compute $DT_A(\bar{A}, \bar{B})$ where $DT_A(A, B) = \partial T_A / \partial A$ and determine when $A \rightarrow \bar{A} = T_A(\bar{A}, \bar{B})$.

3. Having established stability of beliefs B under learning, we compute $DT_C(\bar{B}, \bar{C})$ where $DT_C(B, C) = \partial T_C / \partial C$ and determine when $C \to \bar{C} = T_C(\bar{B}, \bar{C})$.

To solve for $DT_B(\bar{B})$, we linearize $T_B(B)$ at the REE and vectorize the resulting equation. We then use the following identification rule: if $vec(dT_B) = Avec(dB)$ then $A = DT_B(B)$, where $dB = (db(1)\ db(2) \cdots db(S))$ and dT_B is the linearized system of equations. Using the rule: $d(F(X)^{-1}) = -F(X)^{-1}(dF)F(X)^{-1}$, we obtain the following linearization of $T_B(B)$:

$$
\begin{aligned}
dT_B &= \begin{pmatrix} (\Xi(1,B)^{-1}(\sum_{j=1}^{S} p_{1j} M(1,j) db(j)) \Xi(1,B)^{-1} N(1))' \\ (\Xi(2,B)^{-1}(\sum_{j=1}^{S} p_{2j} M(2,j) db(j)) \Xi(2,B)^{-1} N(2))' \\ \vdots \\ (\Xi(S,B)^{-1}(\sum_{j=1}^{S} p_{Sj} M(S,j) db(j)) \Xi(S,B)^{-1} N(S))' \end{pmatrix}' \\
&= \Xi(1,B)^{-1} M(1,1) p_{11} (dB) \begin{pmatrix} \Xi(1,B)^{-1} N(1) & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & & & 0_n \end{pmatrix} \\
&+ \Xi(1,B)^{-1} M(1,2) p_{12} (dB) \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ \Xi(1,B)^{-1} N(1) & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & & & 0_n \end{pmatrix} \\
&+ \cdots \\
&+ \Xi(1,B)^{-1} M(1,S) p_{1S} (dB) \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ \Xi(1,B)^{-1} N(1) & & & 0_n \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \Xi(2,B)^{-1}M(2,1)p_{21}(dB)\begin{pmatrix} 0_n & \Xi(2,B)^{-1}N(2) & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & & & 0_n \end{pmatrix} \\
&+ \quad \Xi(2,B)^{-1}M(2,2)p_{22}(dB)\begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & \Xi(2,B)^{-1}N(2) & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & & & 0_n \end{pmatrix} \\
&+ \quad \cdots \\
&+ \quad \Xi(2,B)^{-1}M(2,S)p_{2S}(dB)\begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & \Xi(2,B)^{-1}N(2) & & 0_n \end{pmatrix} \\
&+ \quad \cdots \\
&+ \quad \Xi(S,B)^{-1}M(S,S)p_{SS}(dB)\begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & & \Xi(S,B)^{-1}N(S) \end{pmatrix}
\end{aligned}
$$

Using the rule $vec(ABC) = C' \otimes Avec(B)$, and the identification rule, we obtain:

$$
\begin{aligned}
DT_B(B) &= \begin{pmatrix} (\Xi(1,B)^{-1}N(1))' & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(1,B)^{-1}M(1,1)p_{11} \\
&+ \begin{pmatrix} 0_n & (\Xi(1,B)^{-1}N(1))' & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(1,B)^{-1}M(1,2)p_{12} \\
&+ \cdots \\
&+ \begin{pmatrix} 0_n & 0_n & \cdots & (\Xi(1,B)^{-1}N(1))' \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(1,B)^{-1}M(1,S)p_{1S} \\
&+ \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ (\Xi(2,B)^{-1}N(2))' & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(2,B)^{-1}M(2,1)p_{21} \\
&+ \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & (\Xi(2,B)^{-1}N(2))' & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(2,B)^{-1}M(2,2)p_{22} \\
&+ \cdots \\
&+ \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & (\Xi(S,B)^{-1}N(S))' \end{pmatrix} \otimes \Xi(S,B)^{-1}M(S,S)p_{SS}
\end{aligned}
$$

Therefore:

$$DT_B(\bar{B}) = \begin{pmatrix} p_{11}\Omega(1)' \otimes F(1,1) & \cdots & p_{1S}\Omega(1)' \otimes F(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}\Omega(S)' \otimes F(S,1) & \cdots & p_{SS}\Omega(S)' \otimes F(S,S) \end{pmatrix}$$

$$\equiv \Psi_{\Omega' \otimes F}$$

E-stability requires the real parts of $\Psi_{\Omega' \otimes F}$ to be less than one. It is important to note that our derivation of the E-stability conditions hinges on the following: $\Xi(i,\bar{B})^{-1}N(i) = \{I - (\sum_{j=1}^{S} p_{ij}M(i,j)\Omega(j))\}^{-1}N(i) = \{I - E_t(M(i,s_{t+1})\Omega(s_{t+1})\}^{-1}N(i) = \Omega(i)$ and $\Xi(i,\bar{B})^{-1}M(i,j) = \{I - E_t(M(i,s_{t+1})\Omega(s_{t+1})\}^{-1}M(i,j) = F(i,j)$. We now turn to the equation for $A = (a(1)'\ a(2)' \cdots a(S)')'$:

$$T_A(A,B) = \begin{pmatrix} \Xi(1,B)^{-1}(\sum_{j=1}^{S} p_{1j}M(1,j)a(j)) \\ \Xi(2,B)^{-1}(\sum_{j=1}^{S} p_{2j}M(2,j)a(j)) \\ \vdots \\ \Xi(S,B)^{-1}(\sum_{j=1}^{S} p_{Sj}M(S,j)a(j)) \end{pmatrix}$$

$$= \begin{pmatrix} p_{11}\Xi(1,B)^{-1}M(1,1) & \cdots & p_{1S}\Xi(1,B)^{-1}M(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}\Xi(S,B)^{-1}M(S,1) & \cdots & p_{SS}\Xi(S,B)^{-1}M(S,S) \end{pmatrix} A$$

Using the same methods as before we obtain the following Jacobian evaluated at the REE where $\bar{A} = 0_{Sn\times 1}$:

$$DT_A(\bar{A},\bar{B}) = \begin{pmatrix} p_{11}F(1,1) & p_{12}F(1,2) & \dots & p_{1S}F(1,S) \\ p_{21}F(2,1) & p_{22}F(2,2) & \dots & p_{2S}F(2,S) \\ \vdots & & \ddots & \vdots \\ p_{S1}F(S,1) & p_{S2}F(S,2) & \dots & p_{SS}F(S,S) \end{pmatrix} \equiv \Psi_F$$

E-stability requires the real parts of Ψ_F to be less than one. Finally, we consider the equation for $C = (c(1)'\ c(2)' \cdots c(S)')'$:

$$T_C(B,C) \quad = \quad \begin{pmatrix} (\Xi(1,B)^{-1}(\sum_{j=1}^{S} p_{1j}M(1,j)c(j)\rho(j) + Q(1)))' \\ (\Xi(2,B)^{-1}(\sum_{j=1}^{S} p_{2j}M(2,J)c(j)\rho(j) + Q(2)))' \\ \vdots \\ (\Xi(S,B)^{-1}(\sum_{j=1}^{S} p_{Sj}M(S,j)c(j)\rho(j) + Q(S)))' \end{pmatrix}'$$

Using the same methods as before we obtain the following Jacobian evaluated at the REE where $\bar{C} = (\Gamma(1)\ \Gamma(2) \cdots \Gamma(S))$:

$$\begin{aligned} DT_C(\bar{B},\bar{C}) \quad &= \quad \begin{pmatrix} p_{11}\rho(1)' \otimes F(1,1) & p_{12}\rho(2)' \otimes F(1,2) & \dots & p_{1S}\rho(S)' \otimes F(1,S) \\ p_{21}\rho(1)' \otimes F(2,1) & p_{22}\rho(2)' \otimes F(2,2) & \dots & p_{2S}\rho(S)' \otimes F(2,S) \\ \vdots & & \ddots & \vdots \\ p_{S1}\rho(1)' \otimes F(S,1) & p_{S2}\rho(2)' \otimes F(S,2) & \dots & p_{SS}\rho(S)' \otimes F(S,S) \end{pmatrix} \\ &\equiv \quad \Psi_{\rho' \otimes F} \end{aligned}$$

■

A.3. Proof of Proposition 4

We prove the the determinacy conditions in Proposition 2 are sufficient for the E-stability conditions stated in Proposition 3. First we define the following arbitary $n \times 1$ MSS $S-$state stochastic processes:

$$\begin{aligned} y_{t+1} &= A(s_t, s_{t+1})y_t + D(s_{t+1})\eta^y_{t+1} \\ z_{t+1} &= B(s_t, s_{t+1})z_t + E(s_{t+1})\eta^z_{t+1} \end{aligned}$$

We place no restrictions on $A(s_t, s_{t+1})$ and $B(s_t, s_{t+1})$ except of course that they are conformable. We also define the corresponding matrix functions of $n \times n$ matrices $A(s_t, s_{t+1})$ and $B(s_t, s_{t+1})$:

$$\Psi_{A\otimes B} = \begin{pmatrix} p_{11}A(1,1)\otimes B(1,1) & \dots & p_{1S}A(1,S)\otimes B(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}A(S,1)\otimes B(S,1) & \dots & p_{SS}A(S,S)\otimes B(S,S) \end{pmatrix}$$

$$\bar{\Psi}_{A\otimes B} = \begin{pmatrix} p_{11}A(1,1)\otimes B(1,1) & \dots & p_{S1}A(S,1)\otimes B(S,1) \\ \vdots & \ddots & \vdots \\ p_{1S}A(1,S)\otimes B(1,S) & \dots & p_{SS}A(S,S)\otimes B(S,S) \end{pmatrix}$$

$$\Psi_A = \begin{pmatrix} p_{11}A(1,1) & \dots & p_{1S}A(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}A(S,1) & \dots & p_{SS}A(S,S) \end{pmatrix}$$

$$\bar{\Psi}_A = \begin{pmatrix} p_{11}A(1,1) & \dots & p_{S1}A(S,1) \\ \vdots & \ddots & \vdots \\ p_{1S}A(1,S) & \dots & p_{SS}A(S,S) \end{pmatrix}$$

Theorem 3. The generic process $y_{t+1} = A(s_t, s_{t+1})y_t + D(s_{t+1})\eta^y_{t+1}$ is MSS if and only if $r(\bar{\Psi}_{A\otimes A}) < 1$.

Proof: see Proposition 3.9 of Costa, Fragoso, and Marques (2005). ■

Note that a MSV solution to (8) is an example of such a process with $A(s_t, s_{t+1}) = \Omega(s_{t+1})$.

Our result in Proposition 4 hinges on three assertions:

Assertion 1. If $r(\bar{\Psi}_{G\otimes G}) < 1$ and $r(\Psi_{F\otimes F}) < 1$ then $r(\Psi_{G'\otimes F}) < 1$ where

$$\Psi_{G'\otimes F} = \begin{pmatrix} p_{11}G(1,1)'\otimes F(1,1) & \dots & p_{1S}G(1,S)'\otimes F(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}G(S,1)'\otimes F(S,1) & \dots & p_{SS}G(S,S)'\otimes F(S,S) \end{pmatrix}$$

Proof: see Appendix C. Proof of Lemma 1 in Cho (2016). ■

Assertion 2. If $r(\bar{\Psi}_{A\otimes A}) = r(\Psi_{A'\otimes A'}) < 1$ then $r(\bar{\Psi}_{A}) = r(\Psi_{A'}) < 1$.

Proof: see Proposition 3.6 in Costa, Fragoso, and Marques (2005). ■

Assertion 3. $x_t = \Omega(s_t)x_{t-1}$ is a MSS process if and only if $x_t = \Omega(s_{t-1})x_{t-1}$ is a MSS process.

Proof: From Theorem 3, $x_t = \Omega(s_t)x_{t-1}$ is MSS if and only if:

$$r\left((\oplus_{j=1}^{S}\Omega(j)\otimes\Omega(j))(P'\otimes I_{n^2})\right) < 1$$

where $\oplus_{j=1}^{S}\Omega(j)\otimes\Omega(j) = diag(\Omega(1)\otimes\Omega(1), \cdots, \Omega(S)\otimes\Omega(S))$. Since:

$$r\left((\oplus_{j=1}^{S}\Omega(j)\otimes\Omega(j))(P'\otimes I_{n^2})\right) = r\left((P'\otimes I_{n^2})(\oplus_{j=1}^{S}\Omega(j)\otimes\Omega(j))\right)$$

and $r\left((P'\otimes I_{n^2})(\oplus_{j=1}^{S}\Omega(j)\otimes\Omega(j))\right) < 1$ if and only if $x_t = \Omega(s_{t-1})x_{t-1}$ is MSS, $x_t = \Omega(s_{t-1})x_{t-1}$ is MSS if and only if $x_t = \Omega(s_t)x_{t-1}$ is MSS. ■

We can equivalently state Proposition 4 as follows: if (1) $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) < 1$ and (2) $r(\Psi_{F^1\otimes F^1}) < 1$[22] the real parts of the eigenvalues of the following matrices are less than one: (i) $\Psi_{\Omega^{1\prime}\otimes F^1}$; (ii) Ψ_{F^1}; (iii) $\bar{\Psi}_{\rho'\otimes F^1}$. As in the main text, Ω^1 denotes the MOD solution.

We prove Proposition 4 as follows. First, determinacy conditions (1) and (2) imply $r(\Psi_{\Omega^{1\prime}\otimes F^1}) < 1$ by Assertion 1 and Assertion 3. Second, we assume that u_t is MSS such that $r(\bar{\Psi}_{\rho\otimes\rho}) < 1$. By Assertion 1, $r(\bar{\Psi}_{\rho\otimes\rho}) < 1$ and determinacy condition (2) imply $r(\bar{\Psi}_{\rho'\otimes F^1}) < 1$. Determinacy condition (2) implies $r(\Psi_{F^1}) < 1$ by Assertion 2.

[22] Again, we abstract from the case $r(\Psi_{F^1\otimes F^1}) = 1$

A.4. Model Calibrations

Table 1: Univariate Model Calibrations

Fig.	$M(1)$	$M(2)$	$N(1)$	$N(2)$	p_{11}	p_{22}
Fig. 3	0.75	0.6	-0.1	0.2	0.9	0.9
Fig. 4	1.6	0.6	-0.1	0.2	0.9	0.9

Table 2: New Keynesian Model Calibrations

Fig.	$\phi(1)$	$\phi(2)$	$\phi_y(1)$	$\phi_y(2)$	$\rho(1)$	$\rho(2)$
Fig. 2	–	–	0	0	–	–
Fig. 5	1.5	0.99	0	0	0.8	0.5
Fig. 6	1.5	0.825	0	0	0.8	0.5
Fig. 7, Par. 1	1.5	0.825	0	0	0.8	0.5
Fig. 7, Par. 2	1.5	0.8	0	0	0.8	0.5
Fig. 7, Par. 3	1.5	0	0	0	0.8	0.5

For Figures 2, 5-7: $\beta = 0.99$, $\sigma = 2$, $\kappa = .1$, $p_{11} = p_{22} = 0.95$ in each calibration.

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