E-stability vis-à-vis Determinacy in Regime-Switching Models

Nigel McClung*

2 July 2020
(Link to Most Recent Draft)

Abstract

This paper examines E-stability, determinacy, and indeterminacy in a general class of regime-switching models with lagged endogenous variables. Using determinacy conditions from Cho (2016, 2020), our first result extends McCallum (2007) to models with time-varying parameters: the unique mean-square stable equilibrium is E-stable when agents use current information and one-period-ahead decision rules. Further, we address existence and E-stability of non-fundamental solutions, and highlight distinctive properties of E-stable solutions of indeterminate regime-switching models. In particular, we find that indeterminate New Keynesian models with recurring interest rate peg regimes can admit an (Iteratively) E-stable MSV solution. In special cases, the Iterative E-stability conditions coincide with the Long Run Taylor Principle.

JEL Classification: C62, D83, D84, E42, E52

Keywords: Markov-Switching, Determinacy, E-stability, Adaptive Learning

*Research Economist, Bank of Finland. Email: Nigel.McClung@bof.fi. The views expressed in this paper do not reflects the views of the Bank of Finland. I am very grateful to Seonghoon Cho, Yunjong Eo, David Evans, George Evans, Chris Gibbs, Bruce McGough, and Jeremy Piger for helpful discussions and comments. This paper supercedes a previous paper titled "E-stability vis-à-vis Determinacy in Markov-switching DSGE Models."

1 Introduction

Rational expectations (RE) models admit multiple equilibria, and economists frequently use two criteria to select an equilibrium. The first criterion ("determinacy") emphasizes model restrictions that ensure the existence of a unique rational expectations equilibrium (REE). Alternatively, the adaptive learning approach of Evans and Honkapohja (2001), among many others, uses "E-stability" to select equilibria that emerge as the outcome of an econometric learning process involving imperfectly informed agents. These two criteria are distinct, but there is value in understanding the connections between them. If we can isolate conditions under which determinacy and E-stability both obtain, then we can dispense with sometimes burdensome E-stability computations. When the two criteria fail to select the same equilibrium, however, it should complicate our understanding of that equilibrium's reasonableness. For example, we might choose to reject an E-unstable determinate equilibrium on the grounds that nearly-rational agents cannot generate the predicted REE dynamics. Additionally, we might give extra consideration to an E-stable equilibrium of an indeterminate model.

This paper examines connections between determinacy, indeterminacy and E-stability in a general class of Markov-switching rational expectations models with lagged endogenous variables. Our contributions are threefold. First, we demonstrate that a set of tractable conditions for determinacy from Cho (2020) imply the E-stability of the unique mean-square stable rational expectations equilibrium if agents know current endogenous variables and use one-period-ahead rules, such as Euler equations, in their decision-making.[1] This contribution extends McCallum (2007), which finds that determinacy implies E-stability in a general class of linear rational expectations models, to environments with time-varying parameters. Additionally, this result extends the E-stability analysis of Branch, Davig, and McGough (2013),

[1]Throughout this the paper, we use "equilibrium" to refer to a mean-square stable or boundedly stable solution of the RE model under study.

who show that a generalization of the Long Run Taylor Principle–which is distinct from the model's determinacy conditions–implies the E-stability of the MSV solution in a restricted class of models that does not include lagged variables. Thus, this paper is the first to show when determinacy (in the mean-square stable sense) is sufficient for E-stability in regime-switching DSGE models, and importantly, the first paper to study E-stability in a general class of regime-switching economies with lagged endogenous variables.[2]

Second, this paper addresses the existence of E-stable solutions of indeterminate regime-switching DSGE models. We find that E-stable non-fundamental (NF) or "sunspot" equilibria exist whenever an E-stable minimal state variable (MSV) solution also exists. These E-stable NF solutions depend on extraneous exogenous variables, including arbitrary lags of the Markov regime. Surprisingly, we also find that indeterminate regime-switching models can have Iterative E-stable ("IE-stable") solutions when agents have current information and use one-period-ahead decision rules, whereas indeterminate linear DSGE models do not admit IE-stable solutions under analogous assumptions about agents' information sets, decision rules, and subjective forecasting models. It has been argued that IE-stable solutions have special properties: Evans and Guesnerie (1993, 2005) and Guesnerie (2002) associate IE-stability with "eductive" stability, or the ability of rational agents to coordinate on a REE using common knowledge of rationality, and Gibbs and McClung (2019) use IE-stability to determine when DSGE models exhibit forward guidance puzzles in the sense of Del Negro, Giannoni and Patterson (2012) and Carlstrom, Fuerst, and Paustian (2015). Thus, our paper spotlights interesting equilibrium properties that distinguish some E-stable solutions of indeterminate Markov-switching DSGE models from all learnable equilibria of indeterminate linear DSGE models.

Third, we present IE-stable solutions of indeterminate New Keynesian models

[2] As with linear DSGE models, many empirically-rich Markov-switching DSGE models include lagged endogenous variables. E.g. see Bianchi (2012), Bianchi and Ilut (2017), Bianchi and Melosi (2017), Chen, Leeper, and Leith (2018).

with persistent interest rate pegs. If we interpret these interest rate peg regimes as recurring zero lower bound (ZLB) episodes, these exercises furthermore reveal one way to construct a model with stable expectations at the ZLB.[3] In special cases, the model's IE-stability conditions coincide with the Long Run Taylor Principle (LRTP) of Davig and Leeper (2007).

This paper is related to a vast literature that studies the relationship between determinacy and E-stability in linear models, including the above mentioned work by McCallum. Ellison and Pearlman (2011) studies "saddlepath learning", a set of RE-consistent restrictions in agents' learning rules that extend McCallum (2007)'s central result to linear DSGE models with lagged information. They find IE-stable indeterminate solutions when agents are saddlepath learning, but IE-stable indeterminate solutions do not exist in linear models under the more standard assumptions we consider here. Bullard and Eusepi (2014) study determinacy and learnability[4] in linear models under general assumptions about agents' information sets and decision rules. They show that determinacy is not generally sufficient for E-stability; McCallum (2007)'s insight is sensitive to details of the learning specification. This paper examines whether McCallum's insight is robust to DSGE model structure and solution concept.

Relatively little work studies determinacy and E-stability in regime-switching models. Most notably, Branch, Davig, and McGough (2013) studies adaptive learning in a class of *purely* forward looking Markov-switching models (i.e. a model without lagged endogenous variables). We build on their path-breaking work in several dimensions. First, we study a more general model class that allows for lagged endoge-

[3]A large adaptive learning literature associates interest rate pegs and the ZLB with E-instability. E.g. see Howitt (1992), Evans, Guse and Honkapohja (2008), Evans and McGough (2018) and Honkapohja and Mitra (2019), among many others. Mertens and Ravn (2014) and Christiano, Eichenbaum, and Johannsen (2018) both consider learnability of equilibria involving one-time transient ZLB regimes, but our analysis allows for recurring ZLB regimes and considers a wide range of calibrations.

[4]This papers uses "E-stable", "learnable", and "expectationally stable" interchangeably.

nous variables. Thus, this paper's insights are applicable to empirically-rich regime-switching DSGE models that feature endogenous persistence arising from features such as habit formation, inflation indexation, capital accumulation, inertial monetary policy, debt-financed government expenditures, and so on. Second, as argued above, we study the relationship between determinacy and E-stability.[5] Third, we comment on the existence and E-stability of non-fundamental ("sunspot") solutions, whereas Branch, Davig and McGough (2013) provide examples of E-stable solutions under special assumptions we discuss below. Finally, we show that regime-switching learning explains stable expectations in economies with persistent interest rate pegs. Some recent related work includes Ozden and Wouters (2020), which studies models with recurring monetary policy regime change and learning agents who employ misspecified linear forecasting models, and Foerster and Matthes (2020), who study Bayesian learning in regime-switching models.

We also contribute to a literature that studies solutions of Markov-switching DSGE models. Farmer, Waggoner, and Zha (2011), Maih (2015), Cho (2016), and Foerster, Rubio-Ramirez, Waggoner and Zha (2016) provide solution techniques that build on the pioneering works of Davig and Leeper (2007), Svensson and Williams (2007), and Farmer, Waggoner, and Zha (2009, 2010). Cho (2016) and Cho (2020) provide conditions for the uniqueness of mean-square stable solutions of Markov-switching models, while Barthelemy and Marx (2019) provide conditions for the uniqueness of bounded solutions. We not address the learnability of bounded solutions of regime-switching models, but we discuss some connections between our work and the determinacy conditions developed by Barthelemy and Marx (2019).

The paper is organized as follows. Section 2 introduces connections between determinacy and E-stability, with Proposition 1-2 presenting our first contributions. Section 3 examines the learnability of indeterminate solutions; Propositions 3-4 and

[5]Reed (2015) also considers determinacy and E-stability, but only in a class of *purely* forward-looking models.

Corollary 4 present new findings. Section 4 examines the Iterative E-stability of solutions to indeterminate New Keynesian models with recurring passive monetary regimes Section 5 concludes.

2 Determinacy and E-stability

Here we characterize determinacy and E-stability properties of a general class of models. First, we reproduce the main finding of McCallum (2007): a unique equilibrium is always E-stable when agents observe contemporaneous endogenous variables and use one-period-ahead forecasting rules. We then turn to a more general class of Markov-switching DSGE (MS-DSGE) models and show that determinacy conditions in Cho (2016) and Cho (2020) imply E-stability of the MSV solution under assumptions that are analogous to assumptions in McCallum (2007).

2.1 Linear DSGE Models

Research on the relationship between determinacy and E-stability often examines widely-used models of the following form:

$$x_t = M\hat{E}_t x_{t+1} + Nx_{t-1} + Qu_t \tag{1}$$

where x_t is a $n \times 1$ vector of endogenous variables, $\hat{E}_t$ denotes (possibly) non-rational expectations conditioned on time-t information, and u_t is a $m\times 1$ covariance-stationary process that follows:

$$u_t = \rho u_{t-1} + \epsilon_t$$

Following Cho (2016, 2020), we express a rational expectations solution to (1) as a linear combination of a minimal state variable (MSV) solution that depends on x_{t-1}

and u_t and a non-fundamental solution component, w_t, such that:

$$x_t = \Omega x_{t-1} + \Gamma u_t + w_t \tag{2}$$

$$w_t = F E_t w_{t+1} \tag{3}$$

where the coefficient matrices satisfy the following conditions:

$$\Omega = (I_n - M\Omega)^{-1} N \tag{4}$$

$$\Gamma = (I_n - M\Omega)^{-1} Q + F\Gamma\rho$$

$$F = (I_n - M\Omega)^{-1} M \tag{5}$$

We refer to (2) as the MSV solution when $w_t = 0$ for all t. Note that Γ and F are uniquely determined by Ω. Hence, we can index any MSV solution to (1) by the equilibrium coefficient matrix Ω. Similarly, any non-fundamental (NF) solution is a linear combination of a MSV solution and w_t and therefore we can think of NF or "sunspot" solutions as being associated to a corresponding MSV solution.[6] We can further characterize the set of MSV solutions, $\mathcal{S}$, as follows:

$$\mathcal{S} = \{\Omega \in \mathcal{C}^{n\times n} | r(\Omega^1) \leq r(\Omega^2) \leq \ldots \leq r(\Omega^N)\} \tag{6}$$

where $r(A)$ denotes the spectral radius of A and N denotes the number of solutions to the fixed point problem (4). McCallum (2007) refers to Ω^1 as the minimum-of-modulus or MOD solution, and it is defined to be $\Omega^1 \in \mathcal{S}$ such that $r(\Omega^1)$ =min $r(\Omega)$ for all Ω in $\mathcal{S}$. In linear models of the form (1), we may employ a variety of techniques to identify the MOD solution, and its existence and uniqueness can be deduced from various properties of the model's eigenvalue-eigenvector system.[7] We explicitly refer to the MOD solution in this section because a unique equilibrium, when it exists,

[6]Section 3 discusses sunspot representations in greater detail.

[7]For example, see Uhlig (1997), Klein (2000), Sims (2002).

is always a MOD solution. As it turns out, we can decide whether a model (1) is determinate simply by identifying the MOD solution.

For the MOD solution to be the unique stable equilibrium, three things need to be true. First, $r(\Omega^1) < 1$ renders the solution non-explosive.[8] Second, the condition $r(F^1) < 1$, where F^1 is defined as in (5), generates an explosive expectational difference equation for all non-fundamental solutions satisfying (3). Therefore this condition precludes coordination on sunspots of the form (3) in the MOD equilibrium. Finally, $1 \leq r(\Omega^2) \leq \ldots \leq r(\Omega^N)$ ensures that all MSV solutions, except for the MOD solution, are explosive. In principle, these conditions can be verified if one identifies $\mathcal{S}$. Such an exercise may prove costly and is inefficient relative to the familiar routines developed by Blanchard and Kahn (1980), Sims (2002) and other well-known works. McCallum (2007) and Cho (2020) provide the following succinct conditions for determinacy in linear models:

Theorem 1 *Consider the model (1) and suppose the MOD solution, Ω^1, exists and is real. (1) is a determinate model if and only if:*

1. $r(\Omega^1) < 1$
2. $r(F^1) \leq 1$

Proof: See Proposition 3 in Cho (2020). ■

Corollary 1 *The model (1) is indeterminate if $r(F^1) > 1$ and $r(\Omega^1) < 1$. In this case, there may exist other MSV solutions, $\Omega \neq \Omega^1$, such that $r(\Omega) < 1$ and $r(F) > 1$.*

Corollary 1 clarifies that $r(F) > 1$ is necessary for any MSV solution of an indeterminate model (1) satisfying $r(\Omega) < 1$, including the MOD solution $r(\Omega_1) < 1$.[9]

[8] Here we use to “non-explosive” to mean “forward stable” or dynamically stable. We do not use “explosive” or “non-explosive” to refer to expectational stability.

[9] See also Appendix D, which proves that if $r(\Omega_1) < 1$ then $r(F) > 1$ for any MSV solution, Ω, of model (1) where $r(\Omega) < 1$ and Ω is not the MOD solution.

The determinacy conditions in Theorem 1 straightforwardly imply the E-stability of the unique MSV solution (MOD solution) given by Ω^1. To show this, we begin by replacing rational agents with learning agents who believe the economy evolves according to a *perceived* law of motion (PLM):[10]

$$x_t = a + bx_{t-1} + cu_t \tag{7}$$

We assume that all agents in the economy have the same PLM, and observe all contemporaneous variables. Stated precisely:

Assumption A. All agents estimate a PLM of the form (7).

Assumption B. Agents observe all contemporaneous variables at time-t (i.e. x_t, u_t are in agents' time-t information sets).

In what follows, we let $\hat{E}_t x_{t+1}$ denote the subjective expectations formed by the learning agents. Since these agents do not know the objective probability distributions for the model's variables and form subjective expectations using their PLM, we can express this expectations term as:

$$\hat{E}_t x_{t+1} = a + bx_t + c\rho u_t$$

It cannot be assumed *a priori* that agents' PLM coincides with the *actual* law of motion (ALM) that governs the equilibrium dynamics in the economy. The ALM is given by a version of (1) which replaces rational expectations with the aforementioned subjective expectations, yielding:

$$x_t = (I_n - Mb)^{-1} Ma + (I_n - Mb)^{-1} N x_{t-1} + (I_n - Mb)^{-1} (Mc\rho + Q) u_t \tag{8}$$

[10]If agents learn in real-time using PLM, (7), then we may express the PLM as $x_t = a_{t-1} + b_{t-1}x_{t-1} + c_{t-1}u_t$ to denote the fact that agents forecast at time-t using their most recent forecasts of (a, b, c) which depend on time-$t-1$ information. For exposition's sake, we suppress the subscripts in $(a_{t-1}, b_{t-1}, c_{t-1})$.

The ALM implies a mapping from the set of beliefs $\Phi = (a, b, c)'$ to the actual equilibrium coefficients of the model, $T(\Phi)$. This mapping is referred to as the *T-map*. In our model, $T(\Phi) = ((I_n - Mb)^{-1} Ma, (I_n - Mb)^{-1} N, (I_n - Mb)^{-1} (Mc\rho + Q))'$. To derive the ALM and corresponding T-map, we assumed that agents make decisions that depend only on expectations of x_{t+1}. In other words, we assume learning agents use "one-period-ahead" decision rules that could arise, e.g., from the first-order conditions in agents' dynamic optimization problems. The one-period-ahead approach is discussed in Evans and Honkapohja (2001), is extensively used in the adaptive learning literature, and moreover, the rational expectations decision rules assume this one-period-ahead form. However, a large literature advocates for an alternative approach in which learning agents' decisions depend on long-horizon expectations.[11] This paper focuses on settings with one-period-ahead decision rules.

Assumption C. Agents use one-period-ahead decision rules (i.e. the ALM is given by (8)).

If agents' beliefs are consistent with (2), then the ALM becomes (2). That is, if $(a, b, c) = (0_{n\times 1}, \Omega, \Gamma) = \bar{\Phi}'$, then $T(\bar{\Phi}) = \bar{\Phi}$. This is another way of stating that rational agents possess true beliefs about the equilibrium mappings in the economy; rational agents believe in perceived laws of motions that are identical to the actual laws of motion. Of course even if our learning agents "learn" the REE coefficients, they will never truly be rational insofar as they will never learn the economy's structure. That is, the learnability of a REE does not by itself justify the behavioral primitives underlying rational expectations. If agents somehow learn the REE, however, it does suggest that there's something about the economy's structure which guides boundedly rational agents to the easy-to-model outcomes predicted by rational expectations analysis. To better understand those stabilizing aspects of an economy's structure we ask the question: when can agents learn to behave like rational agents? In other

[11] E.g. see Preston (2005), Eusepi and Preston (2011), Evans, Honkapohja and Mitra (2013), and Bullard and Eusepi (2014) for more on the merits of learning with infinite-horizon decision rules.

words, what moves (a, b, c) to $(0_{n\times1}, \Omega, \Gamma)$ and under what conditions will such an evolution in beliefs occur? Evans and Honkapohja (2001) shows that a given REE is "E-stable", or, in other words, attainable as the limiting outcome of a real-time learning process if the E-stability conditions in Theorem 2 are satisfied.

Theorem 2 *Consider model (1), and suppose Assumptions A-C hold. Then a REE, $\bar{\Phi}' = (0_{n\times1}, \Omega, \Gamma)$, is said to be E-stable or stable under learning if the real parts of the following three matrices are less than one:*

1. $\Omega' \otimes F$

2. F

3. $\rho' \otimes F$

Proof: see Appendix A. ■

We are finally in a position to restate the main result from McCallum (2007).

Theorem 3 *Suppose Assumptions A-C hold. If (1) is determinate, then the unique equilibrium is E-stable.*

Proof: Determinacy requires $r(\Omega^1) < 1$ and $r(F^1) < 1$.[12] E-stability conditions 2 and 3 follow immediately from $r(F^1) < 1$ and $r(\rho) < 1$. Finally, $r(\Omega^{1'} \otimes F^1) = r(\Omega^1)r(F^1) < 1$. Hence, E-stability condition 1 follows from determinacy. ■

In sum, McCallum (2007) assesses E-stability and determinacy in models of the form (1) by stating both determinacy and E-stability conditions in terms of matrix functions of the MOD solution coefficients. Thus, the MOD solution concept offers a useful bridge between two of the most important equilibrium selection criteria in macroeconomics. In a class of MS-DSGE models that generalizes (1), Cho (2020)

[12] We abstract from the case $r(F^1) = 1$

proposes a MOD solution concept, and a corresponding method that assesses determinacy in said model class using matrix functions of the MOD solution coefficients. Our contribution then establishes E-stability of the unique equilibrium of a MS-DSGE model by studying properties of the MOD solution. We show this in section 2.2.

2.2 Markov-switching DSGE Models and Main Proposition

This paper examines a general class of Markov-switching DSGE models that assume the form:

$$x_t = E_t(M(s_t, s_{t+1})x_{t+1}) + N(s_t)x_{t-1} + Q(s_t)u_t \tag{9}$$

where x_t is a $n \times 1$ vector of endogenous variables, u_t is a $m \times 1$ vector of exogenous variables that follows

$$u_t = \rho(s_t)u_{t-1} + \epsilon_t$$

where s_t is a S-state Markov Chain, $p_{ij} = Pr(s_{t+1} = j|s_t = i)$ is the (i, j)-th element of the transition probability matrix, P, $\rho(s_t)$ is a diagonal matrix and ϵ_t is a white noise process. By assumption, u_t is a mean-square stable process. In this paper, we use the mean-square stability concept, which is widely-used in the MS-DSGE literature. Intuitively, a n-dimensional discrete-time process, such as the RE solution to (9), is mean-square stable if its first and second moments are well-defined as t goes to infinity. More thorough descriptions of mean-square stability can be found in Costa, Fragoso, and Marques (2005), Farmer, Waggoner, and Zha (2009), and Cho (2016). From Theorem 1 in Cho (2016), any rational expectations solution to (9) can be written as a linear combination of a minimal state variable solution that depends on x_{t-1}, s_t, and u_t and a non-fundamental solution component, w_t, as:

$$x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)u_t + w_t \tag{10}$$

$$w_t = E_t\left(F(s_t, s_{t+1})w_{t+1}\right) \tag{11}$$

$$F(s_t, s_{t+1}) = \left(I_n - \sum_{j=1}^{S} p_{s_t j} M(s_t, j)\Omega(j)\right)^{-1} M(s_t, s_{t+1})$$

Cho (2016) provides a tractable method for obtaining solutions of the form (10), which we refer to as a MSV solution when $w_t = 0$. As before, we can think of any NF solution as being associated to a given MSV solution insofar as any sunspot solution can be cast as a linear combination of a MSV solution and extrinsic process, w_t. Cho (2016) also provides sufficient conditions for the existence of a unique mean-square stable MSV solution and non-existence of stable NF solutions of the form w_t. Finally, Cho (2020) shows that closely-related conditions, which we provide in Theorem 4, are not just sufficient but also necessary for ensuring the existence of a unique mean-square stable equilibrium (10). Cho (2020) furthermore generalizes the MOD solution concept to MS-DSGE models (i.e. the MOD solution is the most mean-square stable MSV solution). The MOD solution of (9), denoted as $\Omega^1(s_t)$ for $s_t = 1, \ldots, S$, serves the same purpose in our analysis as the MOD solution of (1): the MOD tells us everything about a model's determinacy properties. We refer interested readers to those papers, and Theorem 4 restates the determinacy conditions from Cho (2016, 2020).

Theorem 4 *Consider the model (9) and suppose the MOD solution, $\Omega^1(s_t)$, exists and is real-valued. (9) is a determinate model if and only if:*

1. $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) < 1$
2. $r(\Psi_{F^1 \otimes F^1}) \leq 1$

where

$$\bar{\Psi}_{\Omega^1\otimes\Omega^1} = \begin{pmatrix} p_{11}\Omega^1(1)\otimes\Omega^1(1) & \dots & p_{S1}\Omega^1(1)\otimes\Omega^1(1) \\ \vdots & \ddots & \vdots \\ p_{1S}\Omega^1(S)\otimes\Omega^1(S) & \dots & p_{SS}\Omega^1(S)\otimes\Omega^1(S) \end{pmatrix}$$

$$\Psi_{F^1\otimes F^1} = \begin{pmatrix} p_{11}F^1(1,1)\otimes F^1(1,1) & \dots & p_{1S}F^1(1,S)\otimes F^1(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}F^1(S,1)\otimes F^1(S,1) & \dots & p_{SS}F^1(S,S)\otimes F^1(S,S) \end{pmatrix}$$

Proof: see Appendix C in Cho (2020). ■

Corollary 2 *The model (9) is indeterminate if $r(\Psi_{F^1\otimes F^1}) > 1$ and $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) < 1$. In this case, there may exist other MSV solutions, $\Omega(s_t) \neq \Omega(s_t)^1$ for some s_t, such that $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ and $r(\Psi_{F\otimes F}) > 1$.*

Corollary 2 clarifies that $r(\Psi_{F\otimes F}) > 1$ is necessary for any MSV solution of an indeterminate model (9) satisfying $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$, including the MOD solution.[13] Theorem 4 presents the Markov-switching DSGE analogue of the determinacy conditions for the linear model presented in Theorem 1.

We now derive the Markov-switching analogue of the adaptive learning model developed in section 2.1. In our MS-DSGE setting, agents' PLM assumes the form:

$$x_t = a(s_t) + b(s_t)x_{t-1} + c(s_t)u_t \tag{12}$$

Notice that agents use a PLM that shares the functional form of the MSV solution. We also assume that agents observe all contemporaneous variables including the Markov state, s_t, when forming expectations at time t and that agents are homogeneous.

[13] See also Appendix D, which proves that if $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) < 1$ then $r(\Psi_{F\otimes F}) > 1$ for any MSV solution, $\Omega(s_t)$, of model (1) where $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ and $\Omega(s_t)$ is not the MOD solution.

Assumption A'. All agents estimate a PLM of the form (12).

Assumption B'. Agents observe all contemporaneous variables at time-t (i.e. x_t, u_t, s_t are in agents' time-t information sets).

In this environment, subjective expectations, $\hat{E}_t x_{t+1}$, can be expressed as follows:

$$\begin{aligned}\hat{E}_t\left(M(s_t,s_{t+1})x_{t+1}\right) &= \hat{E}(M(s_t,s_{t+1})x_{t+1}|s_t=i;x_t,u_t)\\ &= \sum_{j=1}^{S} p_{ij}M(i,j)\left(a(j)+b(j)x_t+c(j)\rho(j)u_t\right)\end{aligned}$$

Substituting $\hat{E}_t(M(s_t,s_{t+1})x_{t+1})$ into (9) yields the ALM:

$$\begin{aligned}x_t &= \left(I-\sum_{j=1}^{S}p_{ij}M(i,j)b(j)\right)^{-1}\left(\sum_{j=1}^{S}p_{ij}M(i,j)a(j)\right)\\ &+ \left(I-\sum_{j=1}^{S}p_{ij}M(i,j)b(j)\right)^{-1}N(i)x_{t-1}\\ &+ \left(I-\sum_{j=1}^{S}p_{ij}M(i,j)b(j)\right)^{-1}\left(\sum_{j=1}^{S}p_{ij}M(i,j)c(j)\rho(j)+Q(i)\right)u_t \qquad (13)\end{aligned}$$

If we define $B=(b(1)\ b(2)\cdots b(S))$ and $\Xi(i,B)=\left(I-\sum_{j=1}^{S}p_{ij}M(i,j)b(j)\right)$ then the state-contingent T-map becomes:

$$\begin{aligned}a(i) &\to \Xi(i,B)^{-1}\sum_{j=1}^{S}p_{ij}M(i,j)a(j)\\ b(i) &\to \Xi(i,B)^{-1}N(i)\\ c(i) &\to \Xi(i,B)^{-1}\left(\sum_{j=1}^{S}p_{ij}M(i,j)c(j)\rho(j)+Q(i)\right)\end{aligned}$$

As before, we assume that learning agents use one-period-ahead decision rules in order to derive the ALM and associated T-map:

Assumption C'. Agents use one-period-ahead decision rules (i.e. the ALM is given by (13)).

What happens if agents learn to behave like rational agents in a REE, $\Omega(s_t)$ (such that $(a(s_t), b(s_t), c(s_t)) = (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$ for all s_t)? Cho (2016) shows that $\Omega(i) = \Xi(i,\Omega)^{-1}N(i)$ and $\Gamma(i) = \Xi(i,\Omega)^{-1}\left(\sum_{j=1}^{S} p_{ij}M(i,j)\Gamma(j)\rho + Q(i)\right)$ for $i = 1,\ldots,S$ where $\Omega = (\Omega(1),\ldots,\Omega(S))$. It immediately follows that $(a(s_t), b(s_t), c(s_t))$ $= (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$ is a fixed point of the T-map, just as a REE in the model class (1) is always a fixed point of the associated T-map. As in section 2.2., we must grapple with the stability of beliefs around these REE fixed points by asking: when will $(a(s_t), b(s_t), c(s_t)) \to (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$ for all s_t?

To answer this question we assume that agents estimate S *within-regime* linear systems. We specifically focus on algorithms of the form:

$$\Phi(s_t)_{ts} = \Phi(s_t)_{ts-1} + \psi_{ts}R(s_t)_{ts}^{-1}z_t(x_t - \Phi(s_t)'_{ts-1}z_t) \quad (14)$$

$$R(s_t)_{ts} = R(s_t)_{ts-1} + \psi_{ts}(z_t z_t' - R(s_t)_{ts-1}) \quad (15)$$

where $z_t = (1, x'_{t-1}, u'_t)'$ and ts is either the number of realizations of state s_t up until time t, or ts is simply equal to t.[14] (14) is a special case of the recursive *conditional* least squares (RCLS) algorithm developed in LeGland and Mevel (1997) for estimation in environments with hidden Markov states. Specifically, the RCLS converges to (14) if we set $R(s_t) = I$ for all s_t, define ψ_{st} appropriately,[15] and add s_t to agents' time-t information sets. It should also be noted that this algorithm is identical to the recursive least squares (RLS) algorithm, which is the workhorse estimation algorithm in the adaptive learning literature, when $S = 1$. We therefore view this recursive learning scheme as a natural extension of RLS to environments

[14] We define ts in this flexible manner in order to make the learning algorithm more general. Basic results in this paper do not depend on the definition of ts, provided that standard regularity assumptions concerning the asymptotic behavior of the gain parameter are satisfied.

[15] E.g., we let $\psi_{st} = t^{-\alpha}$ where $0 < \alpha \leq 1$

with Markov-switching parameters. That is, our approach attempts to be in keeping with the spirit of Evans and Honkapohja (2001). McClung (2019) and other ongoing work more fully explores the properties of these algorithms in models with hidden states.

Because agents are learning Φ via the recursives algorithms (14) and (15), it is clear that beliefs may only converge to REE (i.e. potential convergence points, $\bar{\Phi}$, must satisfy $\bar{\Phi} = T(\bar{\Phi})$ where T denotes the T-map). We therefore follow Branch, Davig, and McGough (2013) and apply the stochastic approximation approach in Evans and Honkapohja (2001) to our regime-switching environment. That is, we derive E-stability conditions from the T-map under the following assumptions:

Proposition 1 *Consider model (8), and suppose Assumptions A'-C' hold. Then a REE,* $\bar{\Phi}(s_t)' = (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$ *for all* s_t*, is said to be E-stable or stable under learning if the real parts of the following three matrices are less than one:*

1. $\Psi_{\Omega'\otimes F}$
2. Ψ_F
3. $\Psi_{\rho'\otimes F}$

where

$$\Psi_{\Omega'\otimes F} = \begin{pmatrix} p_{11}\Omega(1)'\otimes F(1,1) & \cdots & p_{1S}\Omega(1)'\otimes F(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}\Omega(S)'\otimes F(S,1) & \cdots & p_{SS}\Omega(S)'\otimes F(S,S) \end{pmatrix}$$

$$\Psi_F = \begin{pmatrix} p_{11}F(1,1) & \dots & p_{1S}F(1,S) & \\ \vdots & & \ddots & \vdots \\ p_{S1}F(S,1) & \dots & p_{SS}F(S,S) & \end{pmatrix}$$

$$\Psi_{\rho' \otimes F} = \begin{pmatrix} p_{11}\rho(1)' \otimes F(1,1) & \dots & p_{1S}\rho(S)' \otimes F(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}\rho(1)' \otimes F(S,1) & \dots & p_{SS}\rho(S)' \otimes F(S,S) \end{pmatrix}$$

Proof: see Appendix B. ■

These conditions are the Markov-switching DSGE analogue to the conditions presented in McCallum (2007) and Theorem 2. Unsurprisingly, these conditions are identical to the aforementioned E-stability conditions for linear DSGE models when $S = 1$. Having derived the relevant E-stability conditions, we are now in a position to present Proposition 2, which is the main contribution of this section.

Proposition 2 *Suppose Assumptions A'-C' hold. If (9) is determinate, then the unique equilibrium is E-stable.*

Proof: see Appendix C. ■

In linear DSGE models of the form (1), a variety of popular techniques are used to detect the existence and uniqueness of equilibrium.[16] Generally, it is more challenging to obtain MSV solutions in Markov-switching DSGE models, though popular techniques have been developed by Farmer, Waggoner, and Zha (2011), Maih (2015), Cho (2016), and Foerster, Rubio-Ramirez, Waggoner, and Zha (2016), among others. Applications in the remaining sections of this paper use the forward method in Cho (2016) to obtain MOD solutions.

2.2.1 Barthelemy and Marx (2019)

Barthelemy and Marx (2019) provide necessary and sufficient conditions for determinacy in the bounded stability sense for a related class of MS-DSGE models. While this section argues that the unique *mean-square stable* solution selected by Cho (2016)

[16]See, for example: Blanchard and Kahn (1980), Uhlig (1997), Sims (2002).

and Cho (2020) is E-stable, one could be interested in the E-stability of the unique bounded solution. Barthelemy and Marx (2019) confirms that a unique equilibrium assumes the form (9) (with $w_t = 0$) when it exists, and this means that our E-stability conditions apply. However, their approach also involves the computation of a limit which depends on an infinite number of different Markov state histories, and this fact makes it difficult to analytically characterize the relationship between their determinacy conditions and the E-stability conditions presented in this paper.

Because we have not proved that determinacy in the bounded stability sense implies E-stability, an interesting question remains: can a unique equilibrium in the bounded stability sense (i.e. a MSV solution selected by the Barthelemy and Marx (2019) criterion) be E-unstable under our assumptions about information and decision rules? If such an equilibrium, $\Omega^*(s_t)$, exists, then the following is true: $r^e(\Psi_{\Omega^{*\prime}\otimes F^*}) > 1$ or $r^e(\Psi_{F^*}) > 1$. If $r^e(\Psi_{\Omega^{*\prime}\otimes F}) > 1$ or $r^e(\Psi_{F^*}) > 1$ then $r(\Psi_{\Omega^*\otimes\Omega^*}) > 1$ or $r(\Psi_{F^*\otimes F^*}) > 1$. Hence, an E-unstable unique bounded equilibrium is (a) not mean-square stable; or (b) permits mean-square stable sunspot solutions. Regarding (a), a mean-square stable solution has a well-defined first and second moments, and a mean-square unstable solution does not. In practice, we should expect bounded processes to have well-defined first and second moments, but should we encounter a unique bounded equilibrium that is not mean-square stable, we might consider it a poorly-behaved solution due to its inability to deliver well-defined first and second moments for the model's variables. Similarly regarding (b) we might be worried about sunspot indeterminacy if non-fundamental solutions with well-defined first and second moments exist. Apart from these cases, which may give us reasons to further scrutinize the underlying equilibrium, and which we have not found in practice, the main message of this section holds: when a MS-DSGE model admits a unique equilibrium, it is stable under learning.

3 Indeterminacy and E-stability

Section 2 presents a framework in which E-stability is necessary for determinacy, but it is well-known that E-stability is not sufficient for determinacy.[17] This section studies E-stability of indeterminate equilibria of (1) and of (9), and highlights some distinctive properties of E-stable equilibria of indeterminate regime-switching models.

3.1 Linear DSGE Models

To study the learnability of indeterminate equilibria of (1), which include both MSV solution(s) (i.e. (2) with $w_t = 0$) and non-fundamental (NF) or "sunspot" solutions (i.e. (2) with $w_t \neq 0$), we need to specify w_t. One can represent sunspot solutions in many ways, and we elect to use the "common factor" representation:

$$w_t \quad = \quad \Lambda w_{t-1} + VV'\eta_t \tag{16}$$

where the value of Λ is restricted by the model (1), and model equilibrium under study, (2) (e.g. see Evans and McGough (2005, 2005b)), V is an orthonormal matrix, and η_t is some arbitrary Martingale difference sequence (MDS). This paper studies common factor representations because other popular representations, such as the "general form" representation examined by Evans and McGough (2005, 2005b, 2011), are widely associated with E-instability. We note that the different representations of non-fundamental processes encode different informational assumptions; common-factor representations assume that learning agents observe and consequently condition expectations on w_t. Equations (2) and (16) also make it clear that the common factor representation associates a particular MSV solution to each NF solution. Accordingly, we let $NF_{\bar{\Phi}'}$ denote any dynamically stable NF solution associated to

[17] E.g. McCallum (2007) shows that E-stable solutions exist in indeterminate models of the form (1) when Assumptions A-C hold. Evans and McGough (2005, 2005b) also study some of the indeterminate, E-stable solutions we consider in this section.

MSV REE, $\bar{\Phi}' = (0_{n\times 1}, \Omega, \Gamma)$. Finally, if one NF solution, $NF_{\bar{\Phi}'}$, exists, then infinitely many NF solutions exist because η_t can be any MDS. We have now introduced the bare minimum number of details about NF solutions that we need for our analysis; interested readers can see Lubik and Schorfheide (2003, 2004), Evans and McGough (2005, 2005b), or Cho (2020) for more on the derivation of (16).

From (3), $w_t = FE_t\omega_{t+1}$, which imposes the restriction $w_t = 0$ if the model (1) is determinate (such that $r(F) < 1$). It is also well-known that non-zero, dynamically stable (i.e. $r(\Lambda) < 1$) NF solutions of (1) exist if and only if $r(F) > 1$. To study whether these solutions are stable under learning, we assumes agents use a common factor PLM ("CF-PLM") of the form

$$\text{CF-PLM:} \qquad x_t = a + bx_{t-1} + cu_t + dw_t \tag{17}$$

where it is tacitly assumed that w_t is observed and Λ is known.[18] Following the logic of section 2.1, we can show that $T(\Phi) = ((I_n - Mb)^{-1} Ma,\ (I_n - Mb)^{-1} N,\ (I_n - Mb)^{-1} (Mc\rho + Q), d)'$ where $\Phi = (a, b, c, d)'$. The mapping from beliefs, d, to the ALM given by $T(d) = d$ is a standard result in the adaptive learning literature.[19] Since the T-map is otherwise unaffected by the inclusion of w_t in agents' PLM, we can show that for indeterminate cases, a NF solution is E-stable precisely when the associated MSV solution is also E-stable.[20] This last statement is sensitive to assumptions about decision rules, information, etc.

Theorem 5 *Consider model (1) and REE, $\bar{\Phi}' = (0_{n\times 1}, \Omega, \Gamma)$, and suppose Assumptions A-C hold. If*

[18] If w_t is exogenous and observable, then agents can almost surely learn Λ over time.

[19] It is standard to write $T(d) = d$ because multiples of NF processes are NF processes. E.g. if w_t is a sunspot, then so is $T(d)w_t$.

[20] Appendix A. provides techniques needed to show this result. E-stability obtains if the real parts of the eigenvalues of matrices 1-3 in Theorem 2 and the identity matrix are less than *or equal* to 1. In keeping with the literature, we add the condition "or equal" when sunspots are under study. E.g. see Evans and McGough (2005, 2005b).

1. $r^e(F) < 1 < r(F)$

2. $r^e(\Omega \otimes F) < 1$

3. $r^e(\rho \otimes F) < 1$

then the following is true: (a) (1) is indeterminate and $NF_{\bar{\Phi}'}$ exists; (b) MSV REE, $\bar{\Phi}' = (0_{n\times 1}, \Omega, \Gamma)$, is E-stable; (c) $NF_{\bar{\Phi}'}$ is E-stable if agents use CF-PLM, (17).

Proof: From Corollary 1, indeterminacy requires $r(F) > 1 > r(\Omega)$ for any dynamically stable REE of (1). E-stability of the MSV and NF solutions requires $r^e(\rho' \otimes F) < 1$ $r^e(F) < 1$ and $r^e(\Omega \otimes F) < 1$ by Theorem 2 and the main text. ■

Condition 1 of Theorem 5 is particularly important: E-stability and indeterminacy requires $r^e(F) < 1 < r(F)$. These inequalities hold in two different cases:

> **Case 1.** $0 < r^e(F) < 1$ where the maximum of the eigenvalues of F is complex-valued. In this case, there must be at least two eigenvalues of F greater than one in modulus, which means that the degree of indeterminacy, m, exceeds or equals two. Many indeterminate New Keynesian models features $m = 1$, making this case irrelevant for a class of simple monetary macro models.[21]
>
> **Case 2.** $r^e(F) < -1$. Bullard and Mitra (2002) study this case and find indeterminate REE of New Keynesian models with forward-looking interest rate rules that react strongly to the output gap.

Net of cases 1 and 2, which are arguably uncommon in applied work, a dynamically stable REE of (1) is E-stable if and only if $r(F) < 1$ and $r(\Omega \otimes F) < 1$.[22] The conditions $r(\Omega \otimes F) < 1$ and $r(F) < 1$ are known as Iterative E-stability (IE-stability)

[21] E.g. Evans and McGough (2005) find that $m = 2$ in some New Keynesian models, but they do not find learnable equilibria for these cases.

[22] If $r(F) < 1$, $r(\Omega) < 1$, and Ω is real-valued, like any economically reasonable solution, then $r^e(\Omega \otimes F) = r(\Omega \otimes F) < 1$ and $r^e(\rho \otimes F) < 1$.

conditions, and they are a stronger case of E-stability, which require the eigenvalues of F, $\Omega \otimes F$, and $\rho \otimes F$ to be lie inside the unique circle, as opposed to having real parts that are less than one. Some straightforward algebra shows that IE-stability and determinacy are equivalent criteria for the model class (1) under the assumptions stated in Theorems 1-3 and 5.[23]

Corollary 3 *Consider (1) and suppose Assumptions A-C hold. Then an REE,* $\bar{\Phi}' = (0_{n\times 1}, \Omega, \Gamma)$*, is Iteratively E-stable if and only if (1) is determinate.*

IE-stable REE possess some important economic properties. First, Evans and Guesnerie (1992, 2005) and Guesnerie (2002) associate IE-stability with REE that are "eductively" stable, or rationalizable as the outcome a mental learning process in which rational agents coordinate an equilibrium using common knowledge considerations alone. Second, Gibbs and McClung (2019) show that all IE-unstable MSV REE of (1) are prone to forward guidance puzzles a la Del Negro, Giannoni and Patterson (2015), while all IE-stable REE are not. Theorem 5 and Corollary 1 should therefore raise concerns that any E-stable REE of an indeterminate model (1) is eductively unstable (i.e. not rationalizable), and prone to forward guidance puzzles.

3.2 Markov-switching DSGE Models

Now consider (9). As before, we are interested in studying the learnability of indeterminate equilibria, which include MSV solutions (i.e. (10) with $w_t = 0$) and NF solutions associated to these MSV solutions (i.e. (10) with $w_t \neq 0$), and so we study common factor representations following Farmer, Waggoner, and Zha (2009) and Cho

[23]We note that Ellison and Pearlman (2011) find IE-stable equilibria of indeterminate linear models under similar informational assumptions. However, their results only hold when agents are *saddlepath learning*, as explained in the introduction. This paper, and most adaptive learning papers, assume subjective forecasting models share a functional form with the VAR representation of the MSV solution ("MSV learning").

(2016, 2020):[24]

$$w_t = \Lambda(s_{q,t-1}, s_{q,t})w_{t-1} + \eta_t \tag{18}$$

where $s_{q,t} = (s_{t-q+1}, \ldots, s_t)$ for some $q \geq 1$, η_t is an arbitrary MDS, and as in the linear DSGE framework, the value of $\Lambda(s_{q,t-1}, s_{q,t})$ depends on the model (9) and model equilibrium (10) under study (e.g. see Farmer, Waggoner and Zha (2009) and Cho (2020)). From (10) and (18), the common factor representation of a NF solution associates a particular MSV solution to each NF solution of (9), and we let $NF_{\bar{\Phi}(s_t)'}$ denote any mean-square stable NF solution associated to $\bar{\Phi}(s_t)' = (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$. Since η_t is arbitrary, infinitely many mean-square stable NF solutions, $NF_{\bar{\Phi}(s_t)'}$, exist if one such NF solution exists. Notice that $q \geq 1$ implies an equilibrium solution that is "history-dependent" and depends on an arbitrary number of lags of s_t. Branch, Davig, and McGough (2013) consider special cases of these history-dependent equilibria with $q = 1$, $S = 2$, and $N(s_t) = 0$ for all s_t, but they do not comment on the existence of these or other NF solutions. We expand on their analysis by commenting on the existence and learnability of these NF solutions for any q, S and $N(s_t)$. Concerning their existence, Proposition 1 of Cho (2020) argues that mean-square stable NF solutions of the form (10) and (18 exist if and only if $r(\Psi_{F\otimes F}) > 1$. To study whether these solutions are stable under learning, we impose the PLM:

$$\text{CF-PLM:} \qquad x_t = a(s_{q+1,t}) + b(s_{q+1,t})x_{t-1} + c(s_{q+1,t})u_t + dw_t \tag{19}$$

In the last expression we economize on notation by rewriting $(s_{q,t-1}, s_{q,t})$ as $s_{q+1,t}$ (e.g. if $q = 1$ then $s_{q+1,t} = (s_{t-1}, s_t)$). A few remarks are in order. As in section 3.1, we assume that agents observe w_t and know $\Lambda(s_{q,t-1}, s_t)$ when they compute forecasts. Second, we assume the agents' forecasting models are consistent with the lag structure

[24]We focus on common factor representations in regime-switching cases because we do not find that general form representations are learnable. This echoes findings in the linear cases.

implied by (18), but we note that (19) is overparameterized relative to the solution (10) and (18). Third, we could replace dw_t with $d(s_{q+1,t})w_t$ in the PLM, but this would not affect our main findings. Finally, (19) generalizes the "mean-value" PLM of Branch, Davig and McGough (2013) to models with lagged endogenous variables and general history-dependence (e.g. $q \geq 2$). In short, the PLM (19) is logically consistent with the belief that x_t depends on w_t and $s_{q+1,t}$.

To compute the T-map, let $S_q = S^q$ denote the number of regime histories of length q and let P_q denote that transition matrix associated to these state histories where $P_q(i,j) = Pr(s_{q,t+1} = j | s_{q,t} = i)$. Further, note that $M(s_{q+1,t}, s_{q+1,t+1}) = M(s_t, s_{t+1})$, $N(s_{q+1,t}) = N(s_t)$, $Q(s_{q+1,t}) = Q(s_t)$, $\rho(s_{q+1,t}) = \rho(s_t)$, and define $\Xi(i, B_q) = \left(I - \sum_{j=1}^{S_q} P_q(i,j)M(i,j)b(j)\right)$. Then the T-map is:

$$
\begin{aligned}
a(i) &\rightarrow \Xi(i, B_{q+1})^{-1} \sum_{j=1}^{S_{q+1}} P_{q+1}(i,j)M(i,j)a(j) \\
b(i) &\rightarrow \Xi(i, B_{q+1})^{-1} N(i) \\
c(i) &\rightarrow \Xi(i, B_{q+1})^{-1} \left(\sum_{j=1}^{S_{q+1}} P_{q+1}(i,j)M(i,j)c(j)\rho(j) + Q(i) \right) \\
d &\rightarrow d
\end{aligned}
$$

where, again, $d \rightarrow d$ is based on standard arguments in the learning literature, including in Branch, Davig, and McGough (2013). The subsystem associated to $(a(i), b(i), c(i))$ assumes the same form as the T-map when agents use the MSV PLM (12). Moreover, $\Omega(s_{q,t}) = \Omega(s_t)$ and $F(s_{q,t}, s_{q,t+1}) = F(s_t, s_{t+1})$. This means that the relevant E-stability matrices associated to an NF solution of the form (10) and (11) can be derived in the same fashion as matrices 1-3 in Proposition 1.[25] Appendix E presents these matrices and shows that the NF solution (10) and (18) is E-stable if

[25] As in section 3.1, the identity matrix is a relevant E-stability matrix. In keeping with the literature, we modify the E-stability condition to allow for the real part of the eigenvalues of the relevant matrices to be less than *or equal* to 1 for E-stability to obtain.

the MSV solution associated to (10) is also E-stable.[26]

Proposition 3 *Consider model (9) and MSV REE,* $\bar{\Phi}(s_t)' = (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$*, and suppose Assumptions A'-C' hold. If* $\bar{\Phi}(s_t)'$ *is E-stable then* $NF_{\bar{\Phi}(s_t)'}$ *is E-stable.*

Proof: see Appendix E. ■

We are now in a position to characterize the existence and E-stability of indeterminate solutions of regime-switching models.

Proposition 4 *Consider model (9) and REE,* $\bar{\Phi}(s_t)' = (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$*, and suppose Assumptions A'-C' hold. If*

1. $r^e(\Psi_F) < 1 < r(\Psi_{F\otimes F})$
2. $r^e(\Psi_{\Omega\otimes F}) < 1$
3. $r^e(\Psi_{\rho\otimes F}) < 1$

then the following is true: (a) (9) is indeterminate and $NF_{\bar{\Phi}(s_t)'}$ *exist; (b) REE,* $\bar{\Phi}(s_t)' = (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$*, is E-stable; (c)* $NF_{\bar{\Phi}(s_t)'}$ *is E-stable if agents use CF-PLM, (19).*

Proof: From Corollary 2, indeterminacy requires $r(\Psi_{F\otimes F}) > 1$ for any REE of (9), and Proposition 1 of Cho (2020) shows that at least one mean-square stable NF solution of the form (10) and (18) exists. E-stability requires $r^e(\Psi_{\rho'\otimes F}) < 1$, $r^e(\Psi_F) < 1$ and $r^e(\Psi_{\Omega\otimes F}) < 1$ by Proposition 1. Proposition 3 shows that any mean-square stable NF stable solution associated to a given MSV solution is E-stable if the associated MSV solution is E-stable. ■

[26] We note that this finding does not depend on q. I.e. if the MSV solution is E-stable then a mean-square stable NF solution with arbitrary q is mean-square stable.

Like condition 1 of Theorem 5, condition 1 of Proposition 4 is particularly important: E-stability and indeterminacy requires $r^e(\Psi_F) < 1 < r(\Psi_{F\otimes F})$. These inequalities hold in three different cases:

Case A. $0 < r^e(\Psi_F) < 1 < r(\Psi_F) \leq r(\Psi_{F\otimes F})$ where the maximum of the eigenvalues of Ψ_F is complex-valued.

Case B. $r^e(\Psi_F) < -1$.

Case C. $0 < r^e(\Psi_F) \leq r(\Psi_F) < 1 < r(\Psi_{F\otimes F})$.

Cases A and B are akin to Cases 1 and 2 in the linear model (1), and they perfectly coincide in the special case where $S = 1$. Case C is novel: when $S > 1$, $0 < r^e(\Psi_F) = r(\Psi_F) < 1 < r^e(\Psi_{F\otimes F})$ is possible, as demonstrated by Assertion 2 in Appendix A.3, and also in Costa, Fragoso, and Marques (2005), and Cho (2016). It follows from C, that Iterative E-stability conditions, i.e. $r(\Psi_{\Omega\otimes F}) < 1$ and $r(\Psi_F) < 1$, can be satisfied by a REE of an indeterminate model (9).

Corollary 4 *Consider (9), and suppose Assumptions A'-C' hold. Then Iteratively E-stable REE, $\bar{\Phi}(s_t)' = (0_{n\times 1}, \Omega(s_t), \Gamma(s_t))$, may exist when (9) is indeterminate.*

Gibbs and McClung (2019) cite and use the E-stability conditions derived in this paper to show that IE-stable REE of (9) are not susceptible to forward guidance puzzles. Thus, in indeterminate models (9), but not indeterminate models (1), agents can coordinate on REE that generate well-behaved responses to anticipated structural changes, and can be rationalized as the outcome an eductive learning process. This finding distinguishes some indeterminate equilibria of (9) from all indeterminate equilibria of (1).

Our discussion of E-stability and indeterminacy in regime-switching models builds on Branch, Davig and McGough (2013) in three ways. First, and as in section 2, we derive results that apply to the full class of DSGE models (9), including models with lagged endogenous variables. Second, Proposition 3 states the conditions for the existence and E-stability of NF equilibria, whereas Branch, Davig and McGough (2013) provide examples of E-stable NF equilibria. Third, we prove that indeterminate E-stable REE of (9) can be IE-stable, which is not true for the indeterminate equilibria of (1) under Assumptions A-C.

Figure 1 depicts the relationship between the determinacy properties of models (9) and the E-stability of the MSV solution under the assumptions stated in Proposition 2.

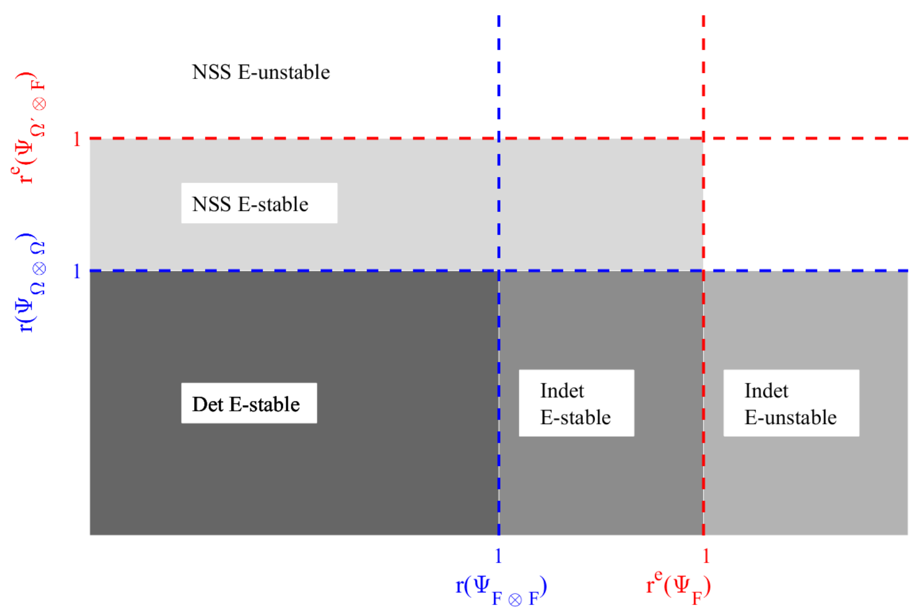


Figure 1: **Existence, Uniqueness and E-stability of Model Equilibria:** Here we suppose that $\Omega(s_t)$ denotes the MOD (minimum-of-modulus) solution (i.e. the most stable solution; see Cho (2020) for more). The MOD solution can be E-stable or E-unstable, and the underlying model is determinate (Det), indeterminate (Indet) or has no stable solutions (NSS). Solutions to the southwest of the red lines are E-stable, while solutions southwest of the blue lines are unique. Thus, determinacy is stronger than E-stability.

4 Application: New Keynesian Model

We introduce a simple New Keynesian model with recurring passive monetary policy regimes to illustrate the new cases discussed in section 3 where E-stability is not sufficient for determinacy. We also show that prolonged, even recurring, ZLB episodes can be conducive to E-stability provided agents account for the transience and recurrence of these episodes in their decision-making (i.e. by employing regime-switching forecasting models of the form (12)).[27] The following describes a New Keynesian model:

$$y_t = \frac{1}{1+\eta}\hat{E}_t y_{t+1} + \frac{\eta}{1+\eta} y_{t-1} - \frac{1-\eta}{\sigma(1+\eta)}(i_t - \hat{E}_t \pi_{t+1}) + u_t^d \tag{20}$$

$$\pi_t = \frac{\beta}{1+\beta\omega}\hat{E}_t \pi_{t+1} + \frac{\kappa}{1+\beta\omega} y_t + \frac{\omega}{1+\beta\omega}\pi_{t-1} + u_t^s \tag{21}$$

where y is the output gap, π is inflation, i is the nominal short-term interest rate, and $\hat{E}_t$ denotes (potentially) non-rational expectations conditioned on time-t information. ω captures inflation indexation in price-setting and η captures external habit formation in household. When $\eta = \omega = 0$ the model collapses to a simple New Keynesian model. By setting $\omega > 0$ and $\eta > 0$ we can study how sources of persistence affect the E-stability properties of policy rules. To complete the model, we pair (20) and (21) with one of the following four policy rules:

$$i_t = \phi_\pi(s_t)\pi_t + \phi_y(s_t) y_t \tag{PR1}$$

$$i_t = \phi_\pi(s_t)\hat{E}_t\pi_{t+1} + \phi_y(s_t)\hat{E}_t y_{t+1} \tag{PR2}$$

$$i_t = \phi_\pi(s_t)\pi_{t-1} + \phi_y(s_t) y_{t-1} \tag{PR3}$$

[27] See a previous version of this paper, McClung (2019b), for evidence from real-time learning simulations that agents can learn the equilibrium coefficients whenever E-stability conditions are satisfied (and only when they are satisfied) and agents use (14) and (15) to update their beliefs.

Equation (PR1) is a standard Taylor rule that sets i in response to contemporaneous inflation and output; (PR2) is a forward-looking rule that sets i in response to expected inflation and output; (PR3) is a backward-looking rule that sets i in response to lagged inflation and output.[28]

The variable s_t follows an exogenous 2-state Markov process (i.e. $s_t \in \{1, 2\}$) with transition matrix, P, where $p_{ij} = Pr(s_{t+1} = j | s_t = i)$. In our discussion of the model here, we treat $s_t = 1$ as a "normal times" regime in which monetary policy is active (e.g. the Taylor Principle, $\phi_\pi > 1$ is satisfied), and $s_t = 2$ represents a passive monetary policy where interest rates are possibly pegged.[29] Finally, we assume $s_t = 2$ is a transient regime (i.e. $p_{22} < 1$) but it recurs if $p_{11} < 1$.

We are interested in the behavior of expectations when $s_t = 2$ as the ZLB, and interest pegs or passive monetary policy more generally, are widely associated with E-instability.[30] In the special case: $p_{22} = 1$ and $s_t = 2$, one could reasonably argue that the correctly-specified PLM is a linear PLM of the form (7). The reason is simple: if $p_{22} = 1$, the economy's structure is time-invariant (i.e. the actual law of motion assumes the form (1)) after the economy enters $s_t = 2$. However, linear PLMs (7) are not consistent with beliefs that (a) $s_t = 2$ is transient (i.e. $p_{22} < 1$); (b) the dynamics of π, y, i differ systematically across regimes, e.g. because monetary policy is passive in one regime and active in the other. If agents *anticipate* a particular type of regime change in the future–i.e. the change from $s_t = 1$ to $s_t = 2$–they should attach *ex ante* probability to that regime change in their forecast. For example, if agents believe $Pr(s_t = 1|s_{t-1} = 1) = p_{11}$, $Pr(s_t = 2|s_{t-1} = 2) = p_{22}$, and that the

[28] In (PR2), $\hat{E}_t$ denotes the fact that the central bank sets i as a function of private sector expectations, which may be non-rational. Hence, we assume that policymakers and private sector agents have identical information sets and expectations formation mechanisms.

[29] To drive interest rates to the ZLB we could augment the model with a switching intercept term as in Bianchi and Melosi (2017) that pegs interest rates below steady state. It is straightforward to show that the corresponding E-stability conditions are unaffected by the inclusion of such an intercept term in the model.

[30] E.g. see Howitt (1992), Evans, Guse and Honkapohja (2008), Evans and McGough (2018), Honkapohja and Mitra (2019), among others.

within-regime dynamics of x_t are described by $x_t = a(1) + b(1)x_{t-1} + c(1)u_t$ when $s_t = 1$ and $x_t = a(2) + b(2)x_{t-1} + c(2)u_t$ when $s_t = 2$, then:

$$\hat{E}_t x_{t+1} = \sum_{j=1}^{2} p_{2j} \left(a(j) + b(j)x_t + c(j)\rho(j)u_t\right)$$

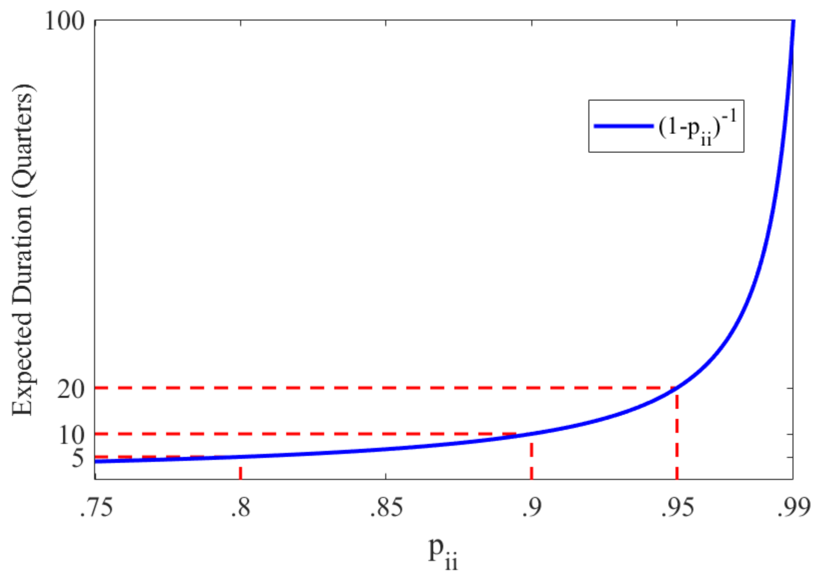
describes a forecast that is consistent with agents' beliefs about the structure of the economy. A linear PLM of the form (7) is never consistent with these beliefs, as a linear PLM does not attach *ex ante* probability to a specific form of regime change–even if agents believe in ongoing structural change and employ a constant gain learning model with high gain parameter.[31] Therefore we assume agents use regime-switching PLMs of the form (12) when $p_{22} < 1$.

What are reasonable values of p_{22} and p_{11}? One approach chooses values p_{ii} for $i = 1, 2$ that deliver empirically plausible or theoretically interesting values of expected regime durations, $(1-p_{ii})^{-1}$. Swanson and Williams (2014) find that the expected duration of the ZLB was between 2-5 quarters prior to the Fed's calendar-based forward guidance in 2011, when the expected duration increased to 10-12 quarters. Kulish, Morley and Robinson (2017) estimate the path of expected durations of the ZLB and obtain similar results. Figure 2 maps expected durations in this range to values of p_{22}. Based on estimated expected durations from Swanson and Williams (2014) and Kulish, Morley and Robinson (2017) and the relationship between expected duration and p_{22} in our model, we restrict attention to values of $p_{22} > .75$.

Figures 3-6 display the E-stability properties of the model when policy regimes are transient and recurring, and agents employ regime-switching PLMs of the form

[31] Many papers study environments in which statistical learning agents track ongoing structural change using a high gain parameter in their estimation algorithm (e.g. see Hollmayr and Matthes (2014)). Importantly, these applications do not assume that agents attach ex ante probability to specific types of regime change in their forecasts, such as a structural change that moves the economy from an ZLB policy regime to an active monetary policy regime, and consequently, these applications model settings in which learning agents believe in ongoing structural change but are agnostic about the type of regime change that may occur in the future.

Figure 2: Expected Regime Duration and Transition Probability



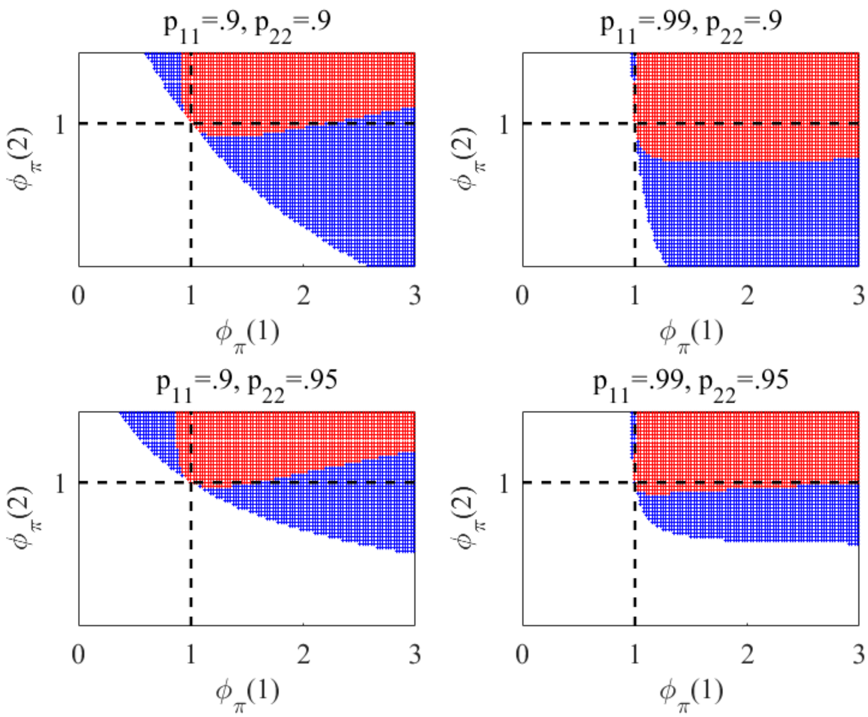
(12) (i.e. Assumptions A'-C' hold). Our benchmark calibration sets $\beta = .995$, $\sigma = 2$, $\kappa = .0164$, and $\phi_y(1) = \phi_y(2) = \omega = \eta = 0$. Our calibration of β and κ is motivated by estimates in Dennis (2009) but we choose a lower, more commonly-used value of σ then the estimated value obtained by Dennis (2009).[32] Finally, we set $\omega = \eta = 0$ so that we can first study E-stability in a simple New Keynesian framework, and we set $\phi_y(1) = \phi_y(2) = 0$ so that the interest rate rules becomes an interest rate peg when $\phi_\pi(1) = \phi_\pi(2) = 0$. We consider alternative calibrations and address robustness concerns below. For both calibrations, we vary the remaining parameters $\phi_\pi(1)$, $\phi_\pi(2)$, p_{11} and p_{22}.

Each panel in Figure 3 shows combinations of $\phi_\pi(1)$ and $\phi_\pi(2)$ that generate determinate, IE-stable and indeterminate, E-stable and indeterminate, or indeterminate and E-unstable models for a given p_{11} and p_{22} under assumptions A'-C' (with $x = (y\ \pi\ i)$). In indeterminate cases, we find one IE-stable MSV solution, if we find an

[32]Dennis (2009) estimates $\theta = .882$ where θ is the fraction of monopolistically competitive firms who do not re-optimize in a given period. Substituting θ into (21) gives $\kappa = (1-\theta)(1-\beta\theta)/(\theta) = .0164$. Dennis (2009) also estimates $\sigma = 5.647$, $\omega = .685$ and $\eta = .824$, but we set σ to a lower, more commonly used value, and $\omega = \eta = 0$ in order to restrict attention to the simple model.

IE-stable solution at all.[33]

Figure 3: Contemporaneous Interest Rate Rule (PR1)



The red area is the determinacy region; the blue area is the IE-stability and indeterminacy region; the white area is the indeterminacy and E-instability region.

Figures 3-5 suggest that while indeterminacy regularly obtains in the model with recurring pegs for values of $p_{22} > .8$, IE-stable REE exist for sufficiently high values of $\phi_\pi(1)$. We also find that backward-looking rules (PR3) promote the largest E-stability and determinacy regions; IE-stable equilibria exist for $p_{22} > .9$ (i.e. expected ZLB durations of at least 10 quarters).

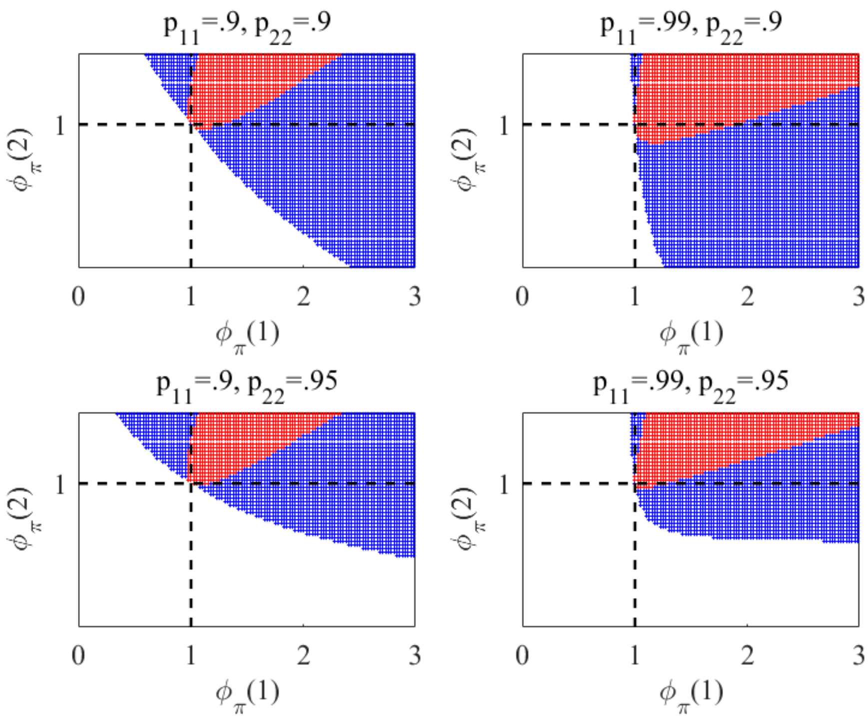
In all cases, determinacy and E-stability regions shrink when β, κ or σ^{-1} increases. Figure 6 displays determinacy and E-stability regions for the model with (PR1) for different values of σ and κ.[34] The top right panel of Figure 6 displays these regions for

[33] This can be explicitly verified using the Gröbner bases approach of Foerster, Rubio-Ramirez, Waggoner, and Zha (2016) to obtain the full set of MSV solutions.

[34] Figure 6 assumes $p_{11}.99$ and $p_{22} = .95$.

a calibration modeled after estimates in Dennis (2009). For this particular calibration, IE-stable solutions obtain when $p_{11} = .99$ and $p_{22} = .95$ (i.e. the expected duration of ZLB regimes is 20 quarters).

Figure 4: Forward-looking Interest Rate Rule (PR2)

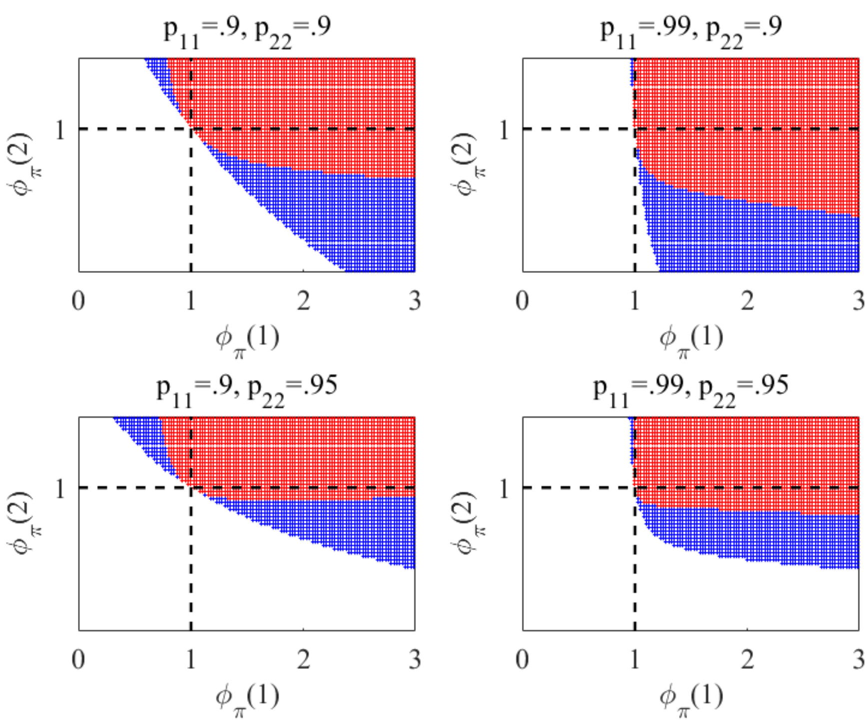


The red area is the determinacy region; the blue area is the IE-stability and indeterminacy region; the white area is the indeterminacy and E-instability region.

Finally, Figure 7 examines E-stability and determinacy across different values of η and ω in the model with contemporaneous rule (PR1). We find that higher values of η and ω tend to enlarge the determinacy regions in $(\phi_\pi(1), \phi_\pi(2))$-space, but shrink the E-stability regions. Thus, accounting for sources of persistence in the model can mitigate the potential for an indeterminate model to admit IE-stable solutions.

We note that the IE-stability condition assigned to an equilibrium of the model (20)-(21) and (PR1) with $\eta = \omega = 0$, which boils down to $r(\Psi_F) < 1$ because there are no lagged endogenous variables, coincides with the Long Run Taylor Principle of Davig and Leeper (2007). Branch, Davig and McGough (2013) also assert a close relationship between E-stability and the LRTP in a class of purely forward-looking models, and Cho (2016) and Barthelemy and Marx (2019) show that the LRTP is substantially weaker than the determinacy conditions developed in their respective papers. Our contribution in this section adds economic significance to their findings: the LRTP, which is a special case of E-stability, tells us when agents can coordinate on the MSV solution via adaptive learning mechanisms and, as it turns out, these conditions are substantially weaker than the conditions under which a unique equilibrium exists. This section's numerical analysis shows that a similar relationship between IE-stability and determinacy exists in models with lagged endogenous variables.

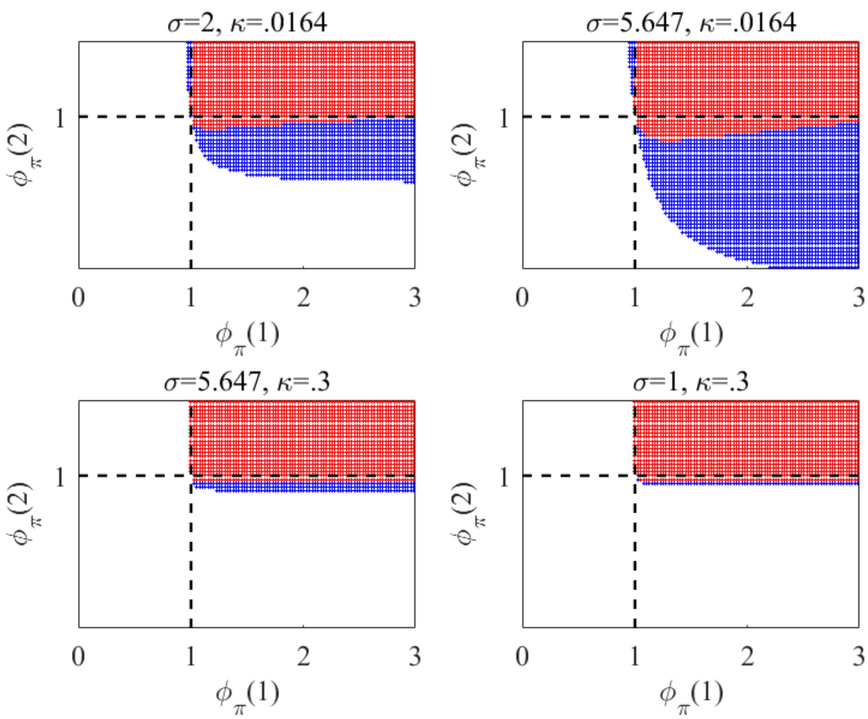
Figure 5: Backward-looking Interest Rate Rule (PR3)



The red area is the determinacy region; the blue area is the IE-stability and indeterminacy region; the white area is the indeterminacy and E-instability region.

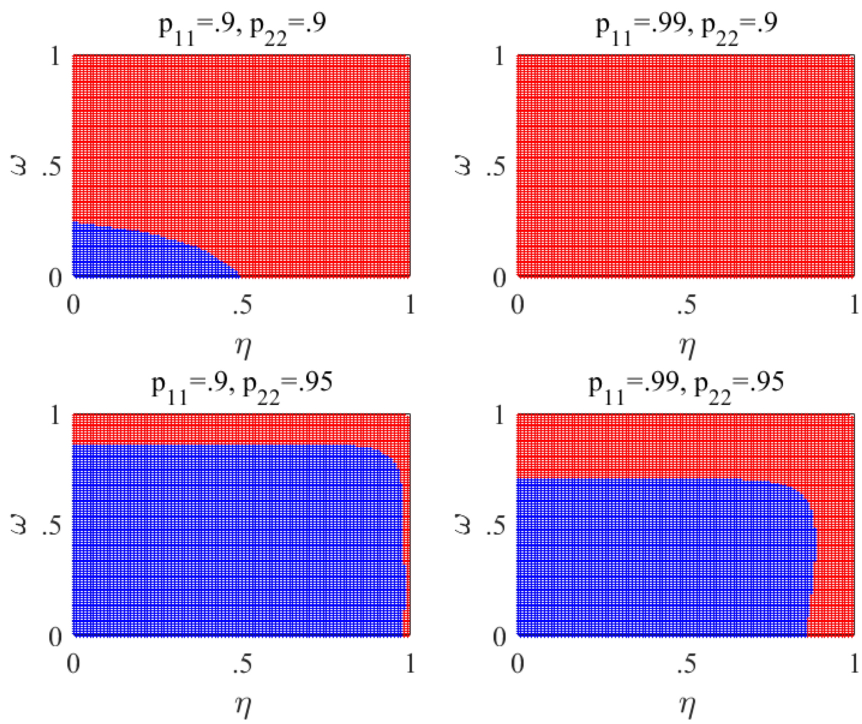
The suite of models described by (20), (21) and (PR1),(PR2), or(PR3), nests

Figure 6: Robustness with Contemporaneous Rule (PR1); $p_{11} = .99$ and $p_{22} = .95$.



The red area is the determinacy region; the blue area is the IE-stability and indeterminacy region; the white area is the indeterminacy and E-instability region. Each panel assumes $p_{11} = .99$ and $p_{22} = .95$. All other parameters are set to benchmark values.

Figure 7: Robustness: Indexation and Habit Persistence



The red area is the determinacy region; the blue area is the IE-stability and indeterminacy region; the white area is the indeterminacy and E-instability region. Each panel assumes $\phi_\pi(1) = 1.5$ and $\phi_\pi(2) = .9$, and that interest rates are given by (PR1). All other parameters are set to benchmark values.

permanent regimes models of the form (1) when $\phi_\pi(s_t) = \phi_\pi$ and $\phi_y(s_t) = \phi_y$ for all s_t. Many papers, notably Bullard and Mitra (2002) and Evans and McGough (2005), have studied the indeterminacy and E-stability properties of these permanent regime models. Two themes from these analyses deserve emphasis:

1. Models of the form (20), (21) and (PR1), (PR2), or (PR3) admit multiple REE, all of which are E-unstable under assumptions A-C with $x = (y\ \pi\ i)'$ or $x = (y\ \pi)$', when monetary policy is characterized by a permanent interest rate peg regime ($\phi_\pi(s_t) = \phi_y(s_t) = 0$ for all s_t). To see this, note that (PR1), (PR2), and (PR3) are identical when $\phi_\pi(s_t) = \phi_y(s_t) = 0$. It follows that

$$r^e(F) = r^e \begin{pmatrix} 1 & \sigma^{-1} \\ \kappa & \kappa\sigma^{-1} + \beta \end{pmatrix} = r(F) > 1$$

   provided $\kappa > 0$ and $\sigma > 0$. This shows that simple New Keynesian models of the form (1) cannot support E-stable REE with interest rate peg regimes.

2. If a model of the form (20), (21) and (PR1), (PR2), or (PR3) admits multiple REE then no REE of the model is IE-stable. This follows from Theorem 5.

Hence, our regime-switching model of adaptive learning helps to explain stability at the ZLB. Under rational expectations these ZLB regimes are subject to extrinsic sunspot volatility, and in standard models of adaptive learning, the ZLB regime is also E-unstable.

5 Conclusion

This paper studies determinacy, indeterminacy and E-stability in a broad class of Markov-switching rational expectations models. We first extend the seminal finding of McCallum (2007), that the unique REE of a linear model is learnable, to regime-switching models. Then we characterize the conditions under which indeterminate REE of regime-switching models are E-stable and point to new examples of indeterminate E-stable REE not previously studied in the literature. In particular, linear DSGE models do not admit Iterative E-stable (IE-stable) solutions of indeterminate models under the standard assumptions we consider, despite the fact that IE-stable solutions exist in indeterminate regime-switching DSGE models, including in simple New Keynesian models with recurring, transient interest rate peg regimes. If we interpret these exogenous interest rate regimes as zero lower bound (ZLB) regimes, these exercises furthermore reveal how to construct a model with stable expectations at the ZLB.

Future research could examine whether estimated regime-switching DSGE models deliver a unique, E-stable equilibrium. Cho and Moreno (2019), McClung (2019) show that some estimated regime-switching DSGE models from the literature are indeterminate, and it remains to be seen whether these models admit E-stable equilibria. Future work may also examine the E-stability properties of regime-switching models under the infinite-horizon learning approach of Preston (2005) and others.

Appendix

Appendix A.. Proof of Proposition 1

In this section we derive the E-stability conditions stated in Proposition 1. More specifically, we derive matrices 1-3 in Proposition 1, and a straightforward application of the E-stability Principle completes the proof. Please note that this proof is also demonstrated in Evans and Honkapohja (2001), p. 238.

We define $\Xi(b) = (I - Mb)$. The state-contingent T-map is given by:

$$\begin{aligned} a &\rightarrow \Xi(b)^{-1} M a \\ b &\rightarrow \Xi(b)^{-1} N \\ c &\rightarrow \Xi(b)^{-1} (M c \rho + Q) \end{aligned}$$

We can express the T-map as $T(a,b,c) = (T_a(a,b), T_b(b), T_c(b,c))$. Define $\Phi = (a,b,c)$, and denote the REE of interest $\bar{\Phi} = (\bar{a}, \bar{b}, \bar{c}) = (0_{n\times1}, \Omega, \Gamma)$. Our task is to compute $DT(\bar{\Phi})$ where $DT(\Phi) = \partial T / \partial \Phi$. By the E-stability Principle, $\bar{\Phi}$ is E-stable if the real parts of the eigenvalues of the matrices comprising $DT(\bar{\Phi})$ are less than one.

Again, it is helpful to consider $T(a,b,c) = (T_a(a,b), T_b(b), T_c(b,c))$. We compute $DT(\bar{\Phi})$ in three stages:

1. Since the system $T_b(b)$ decouples from the rest of the T-map equations, we compute $DT_b(\bar{b})$ where $DT_b(b) = \partial T_b / \partial b$ and establish conditions under which $b \rightarrow \bar{b} = T_b(\bar{b})$.

2. Having established stability of beliefs b under learning, we compute $DT_a(\bar{a}, \bar{b})$ where $DT_a(a,b) = \partial T_a / \partial a$ and determine when $a \rightarrow \bar{a} = T_a(\bar{a}, \bar{b})$.

3. Having established stability of beliefs b under learning, we compute $DT_c(\bar{b}, \bar{c})$

where $DT_c(b,c) = \partial T_c / \partial c$ and determine when $c \to \bar{c} = T_c(\bar{b}, \bar{c})$.

To solve for $DT_b(\bar{b})$, we linearize $T_b(b)$ at the REE and vectorize the resulting equation. We then use the following identification rule: if $vec(dT_b) = Avec(db)$ then $A = DT_b(b)$ where dT_b is the linearized system of equations. We obtain:

$$DT_b(\bar{b}) \quad = \quad \Omega' \otimes F$$

E-stability requires the real parts of $\Omega' \otimes F$ to be less than one. We now turn to the equation for a:

$$T_a(a,b) = \Xi(b)^{-1} Ma$$

Straightforward matrix calculus yields:

$$DT_a(\bar{a}, \bar{b}) = F$$

E-stability requires the real parts of F to be less than one. Finally, we consider the equation for c:

$$T_c(b,c) \quad = \quad \Xi(b)^{-1}(Mc\rho + Q)$$

Using the same methods as before we obtain the following Jacobian evaluated at the REE where $\bar{c} = \Gamma$:

$$DT_c(\bar{b}, \bar{c}) = \rho' \otimes F$$

E-stability therefore requires the real parts of $\rho \otimes F$ to be less than one. ■

Appendix B. Proof of Proposition 1

In this section we derive the E-stability conditions stated in Proposition 3. More specifically, we derive matrices 1-3 in Proposition 3, and a straightforward application of the E-stability Principle completes the proof.

We define $B = (b(1)\ b(2) \cdots b(S))$ and $\Xi(i, B)$ as in section 2.2, and let 0_n denote $n \times n$ matrix of zeros. Additionally, define $A = (a(1)'\ a(2)' \cdots a(S)')'$ and $C = (c(1)\ c(2) \cdots c(S))$. The state-contingent T-map is given by:

$$\begin{aligned} a(i) &\rightarrow \Xi(i,B)^{-1} \sum_{j=1}^{S} p_{ij} M(i,j) a(j) \\ b(i) &\rightarrow \Xi(i,B)^{-1} N(i) \\ c(i) &\rightarrow \Xi(i,B)^{-1} \left(\sum_{j=1}^{S} p_{ij} M(i,j) c(j) \rho(j) + Q(i) \right) \end{aligned}$$

We can express the T-map as $T(A, B, C) = (T_A(A, B), T_B(B), T_C(B, C))$. Define $\Phi = (A, B, C)$, and denote the REE of interest $\bar{\Phi} = (\bar{A}, \bar{B}, \bar{C}) = (0_{nS\times 1}, \Omega, \Gamma)$ where the matrices Ω and Γ collect the state-dependent rational expectations coefficients $\Omega(s_t)$ and $\Gamma(s_t)$, and are conformable to B and C respectively. Our task is to compute $DT(\bar{\Phi})$ where $DT(\Phi) = \partial T / \partial \Phi$. By the E-stability Principle, $\bar{\Phi}$ is E-stable if the real parts of the eigenvalues of the matrices comprising $DT(\bar{\Phi})$ are less than one.

Again, it is helpful to consider $T(A, B, C) = (T_A(A, B), T_B(B), T_C(B, C))$. We compute $DT(\bar{\Phi})$ in three stages:

1. Since the system $T_B(B)$ decouples from the rest of the T-map equations, we compute $DT_B(\bar{B})$ where $DT_B(B) = \partial T_B / \partial B$ and establish conditions under which $B \rightarrow \bar{B} = T_B(\bar{B})$.

2. Having established stability of beliefs B under learning, we compute $DT_A(\bar{A}, \bar{B})$ where $DT_A(A, B) = \partial T_A / \partial A$ and determine when $A \rightarrow \bar{A} = T_A(\bar{A}, \bar{B})$.

3. Having established stability of beliefs B under learning, we compute $DT_C(\bar{B}, \bar{C})$ where $DT_C(B, C) = \partial T_C / \partial C$ and determine when $C \to \bar{C} = T_C(\bar{B}, \bar{C})$.

To solve for $DT_B(\bar{B})$, we linearize $T_B(B)$ at the REE and vectorize the resulting equation. We then use the following identification rule: if $vec(dT_B) = Avec(dB)$ then $A = DT_B(B)$, where $dB = (db(1)\ db(2) \cdots db(S))$ and dT_B is the linearized system of equations. Using the rule: $d(F(X)^{-1}) = -F(X)^{-1}(dF)F(X)^{-1}$, we obtain the following linearization of $T_B(B)$:

$$
\begin{aligned}
dT_B &= \begin{pmatrix} (\Xi(1,B)^{-1}(\sum_{j=1}^{S} p_{1j} M(1,j) db(j))\Xi(1,B)^{-1}N(1))' \\ (\Xi(2,B)^{-1}(\sum_{j=1}^{S} p_{2j} M(2,j) db(j))\Xi(2,B)^{-1}N(2))' \\ \vdots \\ (\Xi(S,B)^{-1}(\sum_{j=1}^{S} p_{Sj} M(S,j) db(j))\Xi(S,B)^{-1}N(S))' \end{pmatrix}' \\
&= \Xi(1,B)^{-1}M(1,1)p_{11}(dB) \begin{pmatrix} \Xi(1,B)^{-1}N(1) & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & & & 0_n \end{pmatrix} \\
&+ \Xi(1,B)^{-1}M(1,2)p_{12}(dB) \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ \Xi(1,B)^{-1}N(1) & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & & & 0_n \end{pmatrix} \\
&+ \cdots \\
&+ \Xi(1,B)^{-1}M(1,S)p_{1S}(dB) \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ \Xi(1,B)^{-1}N(1) & & & 0_n \end{pmatrix}
\end{aligned}
$$

$$+ \quad \Xi(2,B)^{-1}M(2,1)p_{21}(dB)\begin{pmatrix} 0_n & \Xi(2,B)^{-1}N(2) & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & & & 0_n \end{pmatrix}$$

$$+ \quad \Xi(2,B)^{-1}M(2,2)p_{22}(dB)\begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & \Xi(2,B)^{-1}N(2) & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & & & 0_n \end{pmatrix}$$

$$+ \quad \cdots$$

$$+ \quad \Xi(2,B)^{-1}M(2,S)p_{2S}(dB)\begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & \Xi(2,B)^{-1}N(2) & & 0_n \end{pmatrix}$$

$$+ \quad \cdots$$

$$+ \quad \Xi(S,B)^{-1}M(S,S)p_{SS}(dB)\begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & & \Xi(S,B)^{-1}N(S) \end{pmatrix}$$

Using the rule $vec(ABC) = C' \otimes Avec(B)$, and the identification rule, we obtain:

$$
\begin{aligned}
DT_B(B) &= \begin{pmatrix} (\Xi(1,B)^{-1}N(1))' & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(1,B)^{-1}M(1,1)p_{11} \\
&+ \begin{pmatrix} 0_n & (\Xi(1,B)^{-1}N(1))' & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(1,B)^{-1}M(1,2)p_{12} \\
&+ \cdots \\
&+ \begin{pmatrix} 0_n & 0_n & \cdots & (\Xi(1,B)^{-1}N(1))' \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(1,B)^{-1}M(1,S)p_{1S} \\
&+ \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ (\Xi(2,B)^{-1}N(2))' & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(2,B)^{-1}M(2,1)p_{21} \\
&+ \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & (\Xi(2,B)^{-1}N(2))' & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(2,B)^{-1}M(2,2)p_{22} \\
&+ \cdots \\
&+ \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & (\Xi(S,B)^{-1}N(S))' \end{pmatrix} \otimes \Xi(S,B)^{-1}M(S,S)p_{SS}
\end{aligned}
$$

Therefore:

$$DT_B(\bar{B}) = \begin{pmatrix} p_{11}\Omega(1)' \otimes F(1,1) & \cdots & p_{1S}\Omega(1)' \otimes F(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}\Omega(S)' \otimes F(S,1) & \cdots & p_{SS}\Omega(S)' \otimes F(S,S) \end{pmatrix}$$

$$\equiv \Psi_{\Omega' \otimes F}$$

E-stability requires the real parts of $\Psi_{\Omega'\otimes F}$ to be less than one. It is important to note that our derivation of the E-stability conditions hinges on the following: $\Xi(i,\bar{B})^{-1}N(i) = \{I-(\sum_{j=1}^{S} p_{ij}M(i,j)\Omega(j))\}^{-1}N(i) = \{I-E_t(M(i,s_{t+1})\Omega(s_{t+1})\}^{-1}N(i) = \Omega(i)$ and $\Xi(i,\bar{B})^{-1}M(i,j) = \{I - E_t(M(i,s_{t+1})\Omega(s_{t+1})\}^{-1}M(i,j) = F(i,j)$. We now turn to the equation for $A = (a(1)'\ a(2)' \cdots a(S)')'$:

$$T_A(A,B) = \begin{pmatrix} \Xi(1,B)^{-1}(\sum_{j=1}^{S} p_{1j}M(1,j)a(j)) \\ \Xi(2,B)^{-1}(\sum_{j=1}^{S} p_{2j}M(2,j)a(j)) \\ \vdots \\ \Xi(S,B)^{-1}(\sum_{j=1}^{S} p_{Sj}M(S,j)a(j)) \end{pmatrix}$$

$$= \begin{pmatrix} p_{11}\Xi(1,B)^{-1}M(1,1) & \cdots & p_{1S}\Xi(1,B)^{-1}M(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}\Xi(S,B)^{-1}M(S,1) & \cdots & p_{SS}\Xi(S,B)^{-1}M(S,S) \end{pmatrix} A$$

Using the same methods as before we obtain the following Jacobian evaluated at the REE where $\bar{A} = 0_{Sn\times 1}$:

$$DT_A(\bar{A},\bar{B}) = \begin{pmatrix} p_{11}F(1,1) & p_{12}F(1,2) & \dots & p_{1S}F(1,S) \\ p_{21}F(2,1) & p_{22}F(2,2) & \dots & p_{2S}F(2,S) \\ \vdots & & \ddots & \vdots \\ p_{S1}F(S,1) & p_{S2}F(S,2) & \dots & p_{SS}F(S,S) \end{pmatrix} \equiv \Psi_F$$

E-stability requires the real parts of Ψ_F to be less than one. Finally, we consider the equation for $C = (c(1)'\ c(2)' \cdots c(S)')'$:

$$T_C(B,C) \quad = \quad \begin{pmatrix} (\Xi(1,B)^{-1}(\sum_{j=1}^{S} p_{1j}M(1,j)c(j)\rho(j) + Q(1)))' \\ (\Xi(2,B)^{-1}(\sum_{j=1}^{S} p_{2j}M(2,J)c(j)\rho(j) + Q(2)))' \\ \vdots \\ (\Xi(S,B)^{-1}(\sum_{j=1}^{S} p_{Sj}M(S,j)c(j)\rho(j) + Q(S)))' \end{pmatrix}'$$

Using the same methods as before we obtain the following Jacobian evaluated at the REE where $\bar{C} = (\Gamma(1)\ \Gamma(2) \cdots \Gamma(S))$:

$$DT_C(\bar{B},\bar{C}) \quad = \quad \begin{pmatrix} p_{11}\rho(1)' \otimes F(1,1) & p_{12}\rho(2)' \otimes F(1,2) & \dots & p_{1S}\rho(S)' \otimes F(1,S) \\ p_{21}\rho(1)' \otimes F(2,1) & p_{22}\rho(2)' \otimes F(2,2) & \dots & p_{2S}\rho(S)' \otimes F(2,S) \\ \vdots & & \ddots & \vdots \\ p_{S1}\rho(1)' \otimes F(S,1) & p_{S2}\rho(2)' \otimes F(S,2) & \dots & p_{SS}\rho(S)' \otimes F(S,S) \end{pmatrix}$$

$$\equiv \quad \Psi_{\rho' \otimes F}$$

■

Appendix C. Proof of Proposition 2

We prove the the determinacy conditions in Proposition 2 are sufficient for the E-stability conditions stated in Proposition 3. First we define the following arbitary $n \times 1$ MSS $S-$state stochastic processes:

$$\begin{aligned} y_{t+1} &= A(s_t, s_{t+1})y_t + D(s_{t+1})\eta^y_{t+1} \\ z_{t+1} &= B(s_t, s_{t+1})z_t + E(s_{t+1})\eta^z_{t+1} \end{aligned}$$

We place no restrictions on $A(s_t, s_{t+1})$ and $B(s_t, s_{t+1})$ except of course that they are conformable. We also define the corresponding matrix functions of $n \times n$ matrices $A(s_t, s_{t+1})$ and $B(s_t, s_{t+1})$:

$$\Psi_{A\otimes B} = \begin{pmatrix} p_{11}A(1,1)\otimes B(1,1) & \dots & p_{1S}A(1,S)\otimes B(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}A(S,1)\otimes B(S,1) & \dots & p_{SS}A(S,S)\otimes B(S,S) \end{pmatrix}$$

$$\bar{\Psi}_{A\otimes B} = \begin{pmatrix} p_{11}A(1,1)\otimes B(1,1) & \dots & p_{S1}A(S,1)\otimes B(S,1) \\ \vdots & \ddots & \vdots \\ p_{1S}A(1,S)\otimes B(1,S) & \dots & p_{SS}A(S,S)\otimes B(S,S) \end{pmatrix}$$

$$\Psi_{A} = \begin{pmatrix} p_{11}A(1,1) & \dots & p_{1S}A(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}A(S,1) & \dots & p_{SS}A(S,S) \end{pmatrix}$$

$$\bar{\Psi}_{A} = \begin{pmatrix} p_{11}A(1,1) & \dots & p_{S1}A(S,1) \\ \vdots & \ddots & \vdots \\ p_{1S}A(1,S) & \dots & p_{SS}A(S,S) \end{pmatrix}$$

Theorem 3. The generic process $y_{t+1} = A(s_t, s_{t+1})y_t + D(s_{t+1})\eta^y_{t+1}$ is MSS if and only if $r(\bar{\Psi}_{A\otimes A}) < 1$.

Proof: see Proposition 3.9 of Costa, Fragoso, and Marques (2005). ■

Note that a MSV solution to (8) is an example of such a process with $A(s_t, s_{t+1}) = \Omega(s_{t+1})$.

Our result in Proposition 4 hinges on three assertions:

Assertion 1. If $r(\bar{\Psi}_{G\otimes G}) < 1$ and $r(\Psi_{F\otimes F}) < 1$ then $r(\Psi_{G'\otimes F}) < 1$ where

$$\Psi_{G'\otimes F} = \begin{pmatrix} p_{11}G(1,1)'\otimes F(1,1) & \dots & p_{1S}G(1,S)'\otimes F(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}G(S,1)'\otimes F(S,1) & \dots & p_{SS}G(S,S)'\otimes F(S,S) \end{pmatrix}$$

Proof: see Appendix C. Proof of Lemma 1 in Cho (2016). ■

Assertion 2. If $r(\bar{\Psi}_{A\otimes A}) = r(\Psi_{A'\otimes A'}) < 1$ then $r(\bar{\Psi}_A) = r(\Psi_{A'}) < 1$.

Proof: see Proposition 3.6 in Costa, Fragoso, and Marques (2005). ■

Assertion 3. $x_t = \Omega(s_t)x_{t-1}$ is a MSS process if and only if $x_t = \Omega(s_{t-1})x_{t-1}$ is a MSS process.

Proof: From Theorem 3, $x_t = \Omega(s_t)x_{t-1}$ is MSS if and only if:

$$r\left((\oplus_{j=1}^S \Omega(j)\otimes\Omega(j))(P'\otimes I_{n^2})\right) < 1$$

where $\oplus_{j=1}^S \Omega(j)\otimes\Omega(j) = diag(\Omega(1)\otimes\Omega(1), \cdots, \Omega(S)\otimes\Omega(S))$. Since:

$$r\left((\oplus_{j=1}^S \Omega(j)\otimes\Omega(j))(P'\otimes I_{n^2})\right) = r\left((P'\otimes I_{n^2})(\oplus_{j=1}^S \Omega(j)\otimes\Omega(j))\right)$$

and $r\left((P'\otimes I_{n^2})(\oplus_{j=1}^{S}\Omega(j)\otimes\Omega(j))\right) < 1$ if and only if $x_t = \Omega(s_{t-1})x_{t-1}$ is MSS, $x_t = \Omega(s_{t-1})x_{t-1}$ is MSS if and only if $x_t = \Omega(s_t)x_{t-1}$ is MSS. ■

We can equivalently state Proposition 4 as follows: if (1) $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) < 1$ and (2) $r(\Psi_{F^1\otimes F^1}) < 1$[35] the real parts of the eigenvalues of the following matrices are less than one: (i) $\Psi_{\Omega^{1\prime}\otimes F^1}$; (ii) Ψ_{F^1}; (iii) $\bar{\Psi}_{\rho'\otimes F^1}$. As in the main text, Ω^1 denotes the MOD solution.

We prove Proposition 4 as follows. First, determinacy conditions (1) and (2) imply $r(\Psi_{\Omega^{1\prime}\otimes F^1}) < 1$ by Assertion 1 and Assertion 3. Second, we assume that u_t is MSS such that $r(\bar{\Psi}_{\rho\otimes\rho}) < 1$. By Assertion 1, $r(\bar{\Psi}_{\rho\otimes\rho}) < 1$ and determinacy condition (2) imply $r(\bar{\Psi}_{\rho'\otimes F^1}) < 1$. Determinacy condition (2) implies $r(\Psi_{F^1}) < 1$ by Assertion 2.

Appendix D. Properties of Non-MOD Solutions

D.1. Linear Models

Here we show: $r(F) > 1 > r(\Omega)$ for any dynamically stable solution (2) of model (1) other than the MOD solution.[36] First, by definition of the MOD: $r(\Omega) \geq r(\Omega^1)$ where Ω^1 denotes the MOD solution. Suppressing u_t in (1), this allows us to write the solution (2) for x_t as:

$$\begin{aligned} x_t &= \Omega^1 x_{t-1} \\ &= M\Omega^1 x_t + N x_{t-1} \\ &= M(\Omega^1 - \Omega)x_t + M\Omega x_t + N x_{t-1} \\ &= F(\Omega^1 - \Omega)\Omega^1 x_{t-1} + \Omega x_{t-1} \end{aligned}$$

which implies $\Omega^1 - \Omega = F(\Omega^1 - \Omega)\Omega^1$ or $vec(\Omega^1 - \Omega) = ((\Omega^1)' \otimes F))vec(\Omega^1 - \Omega)$. It follows that $r((\Omega^1)' \otimes F)) = r(\Omega^1)r(F) \geq 1$. Since $r(\Omega^1) < 1$, $r(F) > 1 > r(\Omega)$ must

[35] Again, we abstract from the case $r(\Psi_{F^1\otimes F^1}) = 1$

[36] Our argument is based on Appendix B of Cho (2020).

hold for any dynamically stable solution of (1), other than the MOD solution, Ω^1.

D.2. Markov-Switching Models

Here we show: $r(\Psi_{F\otimes F}) > 1 > r(\bar{\Psi}_{\Omega\otimes\Omega})$ for any mean-square stable solution (10) of model (9) other than the MOD solution.[37] First by definition of the MOD: $r(\bar{\Psi}_{\Omega\otimes\Omega}) \geq r(\bar{\Psi}_{\Omega^1\otimes\Omega^1})$ where $\Omega^1(s_t)$ denotes the MOD solution. Suppressing u_t in (9), this allows us to write the solution (2) for x_t as:

$$
\begin{aligned}
x_t &= \Omega^1(s_t)x_{t-1} \\
&= E_t\left(M(s_t,s_{t+1})\Omega^1(s_{t+1})\right)x_t + N(s_t)x_{t-1} \\
&= E_t\left(M(s_t,s_{t+1})(\Omega^1(s_{t+1})-\Omega(s_{t+1}))\right)x_t + E_t\left(M(s_t,s_{t+1})\Omega(s_{t+1})\right)x_t + N(s_t)x_{t-1} \\
&= \left(I_n - E_t\left(M(s_t,s_{t+1})\Omega(s_{t+1})\right)\right)^{-1}\left(E_t\left(M(s_t,s_{t+1})(\Omega^1(s_{t+1})-\Omega(s_{t+1}))\right)\right)x_t \\
&+ \Omega(s_t)x_{t-1} \\
&= \left(E_t\left(F(s_t,s_{t+1})(\Omega^1(s_{t+1})-\Omega(s_{t+1}))\right)\right)\Omega^1(s_t)x_t + \Omega(s_t)x_{t-1}
\end{aligned}
$$

which implies

$$
\Omega^1(s_t) - \Omega(s_t) = \sum_{j=1}^{S} p_{s_t j}F(s_t,j)(\Omega^1(j)-\Omega(j))\Omega^1(s_t) \tag{22}
$$

By vectorizing (22), we have $u = \left(\Psi_{\Omega^{1\prime}\otimes F}\right)u$, where $u = (vec(\Omega^1(1)-\Omega(1))', \ldots, vec(\Omega^1(S)-\Omega(S))')'$ and

$$
\Psi_{\Omega^{1\prime}\otimes F} = \begin{pmatrix} p_{11}\Omega^1(1)'\otimes F(1,1) & \cdots & p_{1S}\Omega^1(1)'\otimes F(1,S) \\ \vdots & \ddots & \vdots \\ p_{S1}\Omega^1(S)'\otimes F(S,1) & \cdots & p_{SS}\Omega^1(S)'\otimes F(S,S) \end{pmatrix}
$$

which implies $r(\Psi_{\Omega^{1\prime}\otimes F}) \geq 1$. From Assertion 1 in Appendix C: $r(\Psi_{\Omega^{1\prime}\otimes F}) \geq 1$ implies

[37] Our argument is based on Appendix B of Cho (2020).

$r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) \geq 1$ or $r(\Psi_{F\otimes F}) \geq 1$. Since $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) < 1$, it follows that $r(\Psi_{F\otimes F}) > 1$.[38] Therefore, $r(\Psi_{F\otimes F}) > 1 > r(\bar{\Psi}_{\Omega\otimes\Omega})$ must hold for any mean-square stable solution of (9) other than the MOD solution.

Appendix E. Proof of Proposition 3

Define $\tilde{q} = q + 1$. Following Appendix B., we can derive the relevant E-stability matrices associated to the NF solution (10) and (18) (i.e. $NF_{\bar{\Phi}(s_t)'}$) from the T-map presented in section 3.2:

$$\tilde{\Psi}_F = \begin{pmatrix} P_{\tilde{q}}(1,1)F(1,1) & \dots & P_{\tilde{q}}(1,S_{\tilde{q}})F(1,S_{\tilde{q}}) \\ \vdots & \ddots & \vdots \\ P_{\tilde{q}}(S_{\tilde{q}},1)F(S_{\tilde{q}},1) & \dots & P_{\tilde{q}}(S_{\tilde{q}},S_{\tilde{q}})F(S_{\tilde{q}},S_{\tilde{q}}) \end{pmatrix} \tag{23}$$

$$\tilde{\Psi}_{\Omega'\otimes F} = \begin{pmatrix} P_{\tilde{q}}(1,1)\Omega(1)'\otimes F(1,1) & \dots & P_{\tilde{q}}(1,S_{\tilde{q}})\Omega(S_{\tilde{q}})'\otimes F(1,S_{\tilde{q}}) \\ \vdots & \ddots & \vdots \\ P_{\tilde{q}}(S_{\tilde{q}},1)\Omega(1)'\otimes F(S_{\tilde{q}},1) & \dots & P_{\tilde{q}}(S_{\tilde{q}},S_{\tilde{q}})\Omega(S_{\tilde{q}})'\otimes F(S_{\tilde{q}},S_{\tilde{q}}) \end{pmatrix} \tag{24}$$

$$\tilde{\Psi}_{\rho'\otimes F} = \begin{pmatrix} P_{\tilde{q}}(1,1)\rho(1)'\otimes F(1,1) & \dots & P_{\tilde{q}}(1,S_{\tilde{q}})\rho(S_{\tilde{q}})'\otimes F(1,S_{\tilde{q}}) \\ \vdots & \ddots & \vdots \\ P_{\tilde{q}}(S_{\tilde{q}},1)\rho(1)'\otimes F(S_{\tilde{q}},1) & \dots & P_{\tilde{q}}(S_{\tilde{q}},S_{\tilde{q}})\rho(S_{\tilde{q}})'\otimes F(S_{\tilde{q}},S_{\tilde{q}}) \end{pmatrix} \tag{25}$$

To compute $P_{\tilde{q}}$ and facilitate a comparison between the E-stability of the model's MSV solution, and E-stability of the NF solution, we index the set of state histories of length q, $\{s_{q,t}\}$, using the lexicographic (lex) order, an increasing numerical order (numbers read left to right).[39] With this ordering of $\{s_{q,t}\}$, and given $\Omega(s_{q,t}) = \Omega(s_t)$, $\rho(s_{q,t}) = \rho(s_t)$, $F(s_{q,t}, s_{q,t+1}) = F(s_t, s_{t+1})$, and $P_q(s_{q,t}, s_{q,t+1}) \in \{p_{ij}\}$, we can recast

[38] We discard the knife-edge case: $r(\Psi_{F\otimes F}) = 1$.

[39] To illustrate, suppose $S = 2$ and $q = 3$. Then $s_{q,t} = (s_{t-2}, s_{t-1}, s_t) \in \{(1,1,1),(1,1,2),(1,2,1),(1,2,2),(2,1,1),(2,1,2),(2,2,1),(2,2,2)\} = S_{q,t}$. The lex order over $S_{q,t}$ gives: $(1,1,1) \prec (1,1,2) \prec (1,2,1) \prec (1,2,2) \prec (2,1,1) \prec (2,1,2) \prec (2,2,1) \prec (2,2,2)$. We now index states according to this ordering: $s_{q,t} = 1$ if $s_{q,t} = (1,1,1)$, $s_{q,t} = 2$ if $s_{q,t} = (1,1,2)$, $s_{q,t} = 3$ if $s_{q,t} = (1,2,1)$, $\dots$,$s_{q,t} = S_q$ if $s_{q,t} = (2,2,2)$.

$\tilde{\Psi}_H$ for $H = F, \rho \otimes F, \Omega \otimes F$ as

$$\tilde{\Psi}_H = i_S \otimes I_q \otimes \hat{\Psi}_H \tag{26}$$

$$\hat{\Psi}_H = \begin{pmatrix} \Psi_{H,1} & 0 & \dots & 0 \\ 0 & \Psi_{H,2} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & \Psi_{H,S} \end{pmatrix} \tag{27}$$

where $q \geq 1$, i_S is an S by 1 vector of ones, $\Psi_{H,i}$ denotes the i-th block of v_H rows of Ψ_H where $v_F = n$, $v_{\rho\otimes F} = mn$ and $v_{\Omega\otimes F} = n^2$. For each H, $rank(\tilde{\Psi}_H) = rank(I_q \otimes \hat{\Psi}_H) \leq Sqv_H$.

To see that E-stability of the MSV solution, $\bar{\Phi}(s_t)'$, is necessary for E-stability of $NF_{\bar{\Phi}(s_t)'}$, let λ_H denote any eigenvalue of Ψ_H with corresponding right eigenvector, μ_H. Then by explicit computation, we can show that $(\mu'_H, \mu'_H, \dots, \mu'_H)'$ is a right eigenvector of $\tilde{\Psi}_H$. Hence, the eigenvalues of Ψ_H are also eigenvalues of $\tilde{\Psi}_H$ which implies $r(\Psi_H) \leq r(\tilde{\Psi}_H)$ for $H = F, \rho \otimes F, \Omega \otimes F$.

To see that E-stability of the MSV solution, $\bar{\Phi}(s_t)'$, is also sufficient for E-stability of $NF_{\bar{\Phi}(s_t)'}$ let $\tilde{\lambda}_H$ denote any eigenvalue of $\tilde{\Psi}_H$ with corresponding left eigenvector, $\tilde{\mu}'_H$. We can write: $\tilde{\mu}'_H = (\tilde{\mu}'_1, \tilde{\mu}'_2, \dots, \tilde{\mu}'_{S^2q})$ where μ_i is v_H times 1 for $i = 1, \dots S^2q$. Through explicit computation, the following holds

$$\Psi'_{H,i}\left(\sum_{k=0}^{S-1} \mu_{Sqk+(j-1)S+i}\right) = \tilde{\lambda}_H \begin{pmatrix} \mu_{(j-1)S^2+(i-1)S+1} \\ \vdots \\ \mu_{(j-1)S^2+(i-1)S+S} \end{pmatrix}$$

for $i = 1, \ldots, S$, $j = 1, \ldots, q$. Summing over i, j and rearranging gives

$$(\Psi_H)' \begin{pmatrix} \sum_{l=0}^{Sq-1} \mu_{Sl+1} \\ \vdots \\ \sum_{l=0}^{Sq-1} \mu_{Sl+S} \end{pmatrix} = \tilde{\lambda}_H \begin{pmatrix} \sum_{l=0}^{Sq-1} \mu_{Sl+1} \\ \vdots \\ \sum_{l=0}^{Sq-1} \mu_{Sl+S} \end{pmatrix}$$

Hence, $\tilde{\lambda}_H$ is also an eigenvalue of Φ_H. It follows that $r(\Psi_H) \geq r(\tilde{\Psi}_H)$ for $H = F, \rho \otimes F, \Omega \otimes F$. ■

Works Referenced

Barthelemy, J., & Marx M. (2019). Monetary policy switching and indeterminacy. *Quantitative Economics 10*, 353-385.

Bianchi, F. (2012). Evolving Monetary/Fiscal Policy Mix in the United States. *American Economic Review Papers and Proceedings*, 101(3), 167-172.

———— , & C. Ilut. (2017). Monetary/Fiscal Policy Mix and Agents' Beliefs. *Review of Economic Dynamics*, 26, 113-139.

Bianchi, F., & Melosi, L. (2017). Escaping the Great Recession. *American Economic Review 107*(4), 1030-1058.

Blanchard, O., & Kahn, C. (1980). Solution of linear difference models under rational expectations. *Econometrica 38*, 1305-1311.

Branch, B., Davig, T., & B. McGough. (2013). Adaptive Learning in Regime-Switching Models. *Macroeconomic Dynamics 17*(5), 998-1022.

Bullard, J., & Eusepi, S. (2014). When Does Determinacy Imply Expectational Stability. *International Economic Review 55*(1), 1-22.

Bullard, J., & Mitra, K. (2002). Learning about monetary policy rules. *Journal of Monetary Economics 49*(6), 1105-1129.

Carlstrom, C-, Fuerst, T. & Paustian, M. (2015). Inflation and output in New Keynesian models with a transient interest rate peg. *Journal of Monetary Economics 76*, 230-243.

Chen, X., E. M. Leeper, & C. Leith. (2018). U.S. Monetary and Fiscal Policy: Conflict or Cooperation? Manuscript.

Cho, S. (2016). Sufficient Conditions for Determinacy in a Class of Markov-Switching Rational Expectations Models. *Review of Economic Dynamics 21*, 182-200.

————. (2020). Determinacy and Classification of Markov-Switching Rational Expectations Models. Available at SSRN: https://ssrn.com/abstract=3229215 or http://dx.doi.org/10.2139/ssrn.3229215.

Cho, S. & Moreno, A. (2019). Has Fiscal Policy Saved the Great Recession?. Available at SSRN: https://ssrn.com/abstract=3447189

Christiano, L., Eichenbaum, M., & B. Johannsen (2018): "Does the New Keynesian Model Have a Uniqueness Problem?" Manuscript.

Costa, O., Fragoso, M., & Marques, R. (2005). *Discrete-Time Markov Jump Linear Systems* Spring, New York: Springer-Verlag London.

Davig, T., & Leeper, E. (2007). Generalizing the Taylor Principle," *American Economic Review 97*(3), 607-635.

Del Negro, M., Giannoni, M., & Patterson, C. (2012). The forward guidance puzzle. FRB of New York Staff Report, (574).

Dennis, R. (2009). Consumption Habits in a New Keynesian Business Cycle Model. *Journal of Money, Credit and Banking 41*(5), 1015-1030.

Ellison, M., & Pearlman, J. (2011). Saddlepath Learning. *Journal of Economic Theory 146*(4), 1500-1519.

Eusepi, S., & Preston, B. (2011a). Expectations, Learning, and Business Cycle Fluctuations. *American Economic Review* 101 (6), 2844-72.

Evans, G., & Guesnerie, R. (1993). Rationalizability, strong rationality and expectation stability. *Games and Economic Behavior 5*, 632-646.

Evans, G., & Guesnerie, R. (2005). Coordination of saddle-path solutions: the eductive viewpoint–linear multivariate models. *Journal of Economic Theory 124*, 202-229.

Evans, G., Guse, E., & S. Honkapohja. (2008). Liquidity traps, learning and stagnation. *European Economic Review 52*(8), 1438-1463.

Evans, G., & Honkapohja, S. (2001). *Learning and Expectations in Macroeconomics.* Princeton, New Jersey: Princeton University Press.

Evans, G., Honkapohja, S., & Mitra, K. 2013. Notes on Agents' Behavioral Rules under Adaptive Learning and Studies of Monetary Policy. In *Macroeconomics* *at the Service of Public Policy*, edited by Thomas J. Sargent and Juoko Vilmunen, 63-79. Oxford and New York: Oxford University Press.

Evans, G., & B. McGough. (2005). Monetary policy,indeterminacy and learning. *Journal of Economic Dynamics and Control*, 29, 1809-1840.

———. (2005b). Stable sunspot solutions in models with predetermined variables. *Journal of Economic Dynamics and Control*, 29, 601-625.

———. (2011). Representations and Sunspot Stability. *Macroeconomic Dynamics*, 15, 80-92.

———. (2018). Interest-Rate Pegs in New Keynesian Models. *Journal of Money, Credit, and Banking*, 50(5), 939-965.

Farmer, R., Waggoner, D., & Zha, T. (2009). Understanding Markov-Switching Rational Expectations Models. *Journal of Economic Theory 144*(5), 1849-1867.

———. (2010). Generalizing the Taylor Principle: A Comment. *American Economic Review 101*(1), 608-617.

———. (2011). Minimal State Variable Solutions to Markov-Switching Rational Expectations Models. *Journal of Economic Dynamics and Control 35*(12), 2150-2166.

Foerster, A., & Matthes, C. 2020. Learning about Regime Change. Federal Reserve Bank of San Francisco Working Paper 2020-15.

Foerster, A., Rubio-Ramirez, J., Waggoner, D., & Zha, T. (2016). Perturbation methods for Markov-switching dynamic stochastic general equilibrium models. *Quantitative Economics 7*(2), 637-669.

Guesnerie, R. (2002). Anchoring economic predictions in common knowledge. *Econometrica 70* (2).

Gibbs, C. & N. McClung (2019). Does my model predict a forward guidance puzzle? Bank of Finland Discussion Paper 19/2029.

Hollmayr, J. & Matthes, C. (2015). Learning about fiscal policy and the effects of policy uncertainty. *Journal of Economic Dynamics and Control* 59, 142-162.

Honkapohja, S., & K. Mitra. (2019). Price level targeting and evolving credibility. *Journal of Monetary Economics*, forthcoming.

Howitt, P. (1992). Interest Rate Control and Nonconvergence to Rational Expectations. *Journal of Political Economy 100*, 776-800.

Klein, P. (2000). Using The Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model. *Journal of Economic Dynamics and Control 24*(8), 1405-1423.

Kulish, M., Morley, J., & Robinson, T. (2017). Estimating DSGE models with zero interest rate policy. *Journal of Monetary Economics*, 88, 35-49.

LeGland, F., & Mevel, L. (1997). Recursive estimation of hidden Markov models. Proc. 36th IEEE Conf. Decision Control, San Diego, CA, Dec. 1997.

Lubik, T., & Schorfheide, F. (2003). Computing sunspot equilibria in linear rational expectations models. *Journal of Economic Dynamics and Control*, 28, 273.285.

————. (2004). Testing for Indeterminacy: An Application to U.S. Monetary Policy. *The American Economic Review*, 94(1), 190-217.

Maih, J. (2015). Efficient perturbation methods for solving regime-switching DSGE models. Working Paper 2015/01, Norges Bank.

McClung, N. (2019). Performance of Simple Interest Rate Rules Subject to Fiscal Policy. Mimeo

McClung, N. (2019b). E-stability vis-á-vis Determinacy in Markov-Switching DSGE Models. Available at SSRN: https://ssrn.com/abstract=3393007

Mertens, K., and M. Ravn. (2014). Fiscal Policy in an Expectations-Driven Liquidity Trap,” *Review of Economic Studies* 81, 1637-1667.

McCallum, B.T. (2007). E-Stability Vis-A-Vis Determinacy Results For A Broad Class Of Linear Rational Expectations Models. *Journal of Economic Dynamics and Control 31*(4), 1376-1391.

Ozden, T., & Wouters, R. (2020). Restricted Perceptions, Regime Switches and the Zero Lower Bound. Mimeo.

Preston, B. (2005). Learning about Monetary Policy Rules when Long-Horizon Expectations Matter. *International Journal of Central Banking 1*(2).

Reed, J.R. (2015). Mean-square Stability and Adaptive Learning in Regime-Switching Models. Manuscript.

Sims, C. (2002). Solving Linear Rational Expectations Models. *Computational Economics 20*(1), 1-20.

Svensson, L. E. O. and Williams, N. (2007). Monetary policy with model uncertainty: Distribution forecast targeting. CEPR Discussion Papers 6331, C.E.P.R.

Swanson, E, and Williams, J. (2014). Measuring the Effect of the Zero Lower Bound on Medium- and Longer-Term Interest Rates. *American Economic Review 104*(10), 3154-3185.

Uhlig, H. (1997). A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily. In R. Marimon and A. Scott (eds.), Computational Methods for the Study of Dynamic Economies (pp.30-61). Oxford, England: Oxford University Press.