Resolving New Keynesian puzzles

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Abstract

New Keynesian macroeconomic models generate well known puzzles when confronted with the zero lower bound (ZLB) on nominal interest rates. Known resolutions of these puzzles focus on deep structural features of markets and agent behavior (e.g., complete markets, rational expectations, etc.) or the behavior of the fiscal authority. We show a simpler explanation and resolution of the puzzles that supersedes those put forward to date. The standard way monetary policy is modeled at the ZLB does not correspond to that which a dual mandate inflation targeter would choose at the ZLB or after liftoff, and it is a radical departure from the policy assumed outside of the ZLB. Employing policy rules that approximate, even loosely, the pre-ZLB objectives of policy while at the ZLB effectively resolves the New Keynesian puzzles. The puzzles, therefore, are best thought of as extreme predictions of implausible monetary policy rather than implausible predictions to plausible monetary policy.

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1 Introduction

While the New Keynesian (NK) model provides an elegant theory of monetary policy during the Great Moderation, when confronted by the zero lower bound (ZLB) on nominal interest rates, it has yielded a litany of puzzles and paradoxes. The most well known is the so-called forward guidance puzzle, which is most simply summarized as, *why do standard quantitative models used by central banks the-world-over explode when we attempt to model real-world policy at the ZLB?*

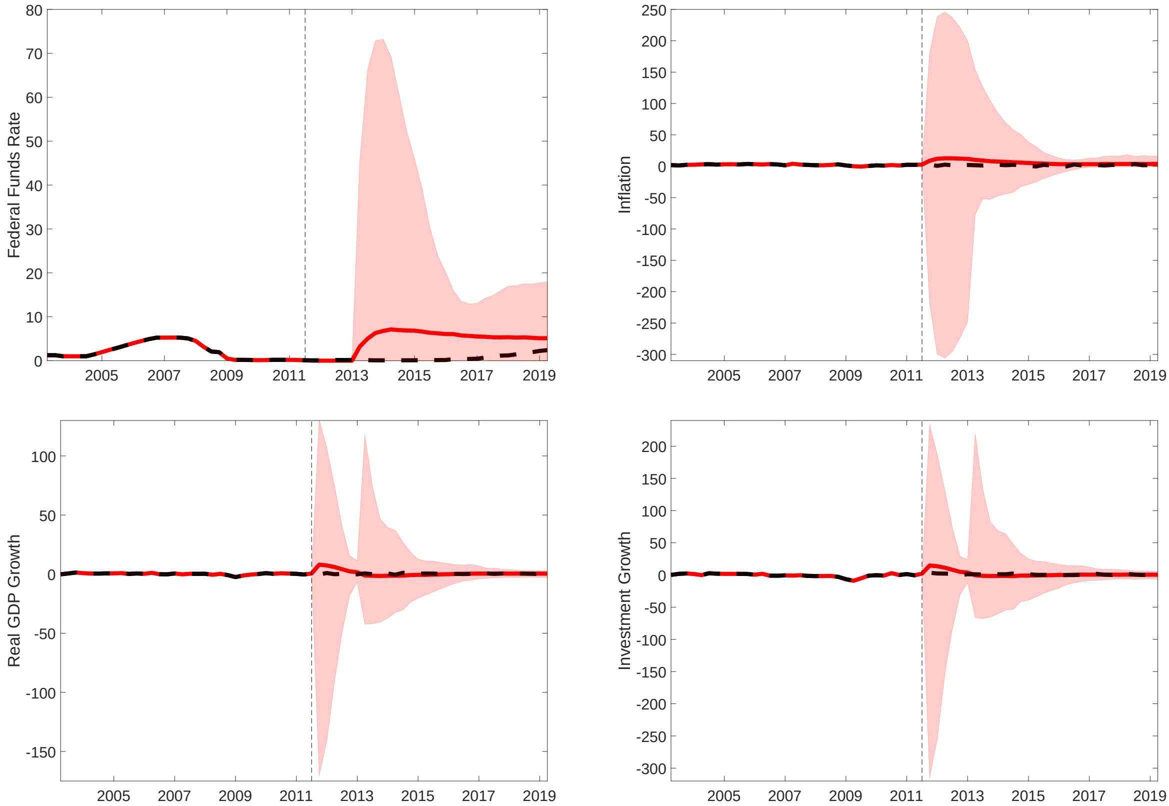
To illustrate what *explode* means in this context, consider the plight of economists in August of 2011 attempting to provide forecasts based on the following FOMC statement:

> *The Committee currently anticipates that economic conditions ... are likely to warrant exceptionally low levels for the federal funds rate at least through mid-2013.*
>
> \- FOMC Statement August, 9^{th} 2011

Figure 1 shows what forecasts of this policy for inflation, real GDP growth, investment, and the Federal Funds look like using the model of Smets and Wouters (2007). The forecasts are constructed using Smets and Wouter's original replication files and data (which ends in 2004) to estimate the posterior distribution of the structural parameters. We then follow Cagliarini and Kulish (2013), Jones (2017), and Kulish and Pagan (2017) to generate forecasts by sampling from the estimated posterior distribution of the parameters while enforcing a fully credible and known policy of zero interest rates for seven quarters. After seven quarters, monetary policy is governed by an occasionally binding constraints algorithm, where the policy rate is the maximum of zero or the interest rate implied by the model's interest rate rule.

Clearly, Figure 1 shows that there is a problem. Indeed, it is not surprising that the first papers documenting the forward guidance puzzle came from within the Federal Reserve System. For example, the work by Del Negro, Giannoni and Patterson (2012) or Carlstrom, Fuerst and Paustian (2015), to our knowledge, are among the first papers to study the exact issue shown in Figure 1. The forward guidance puzzle is also not the only ZLB NK puzzle. There are at least three additional puzzles widely studied in the literature: the fiscal multiplier puzzle (e.g., Farhi and Werning, 2016), the paradox of toil (e.g., Eggertsson, 2010), and the paradox of the flexi-

Figure 1: Calendar-based forward guidance and the forward guidance puzzle



Notes: Out-of-sample forecasts of the August, 9^{th} 2011 Federal Reserve calendar-based forward guidance promise using the model of Smets and Wouters (2007). All variables show Q/Q forecasts. Inflation and the Federal Funds Rates are annualized.

bility (e.g., Eggertsson and Krugman, 2012 or Kiley, 2016). Like the forward guidance puzzle, these puzzles produce predictions at odds with economic intuition and empirical evidence.

We, of course, agree that these puzzles indicate a problem with New Keynesian models. The problem though is not with the underlying representative agent assumption, rational expectations, or the complete market assumptions, which are often faulted in the literature. The problem lies in our convention of modeling monetary policy in a reduced form way using Taylor (1993) type rules.

In nearly all modern models of monetary policy, the preferences and actions of the central bank are reduced to rules of the following form

$$i_t = (1 - \rho_i)\bar{r} + \rho i_{t-1} + (1 - \rho)(\phi_\pi \pi_t + \phi_y y_t), \tag{1}$$

where i_t is the policy rate chosen by the central bank, π_t is inflation (assuming a zero percent inflation target), y_t is some measure of real activity (usually the output gap), and $\bar{r}$ is the steady state real interest in the economy. The parameter ϕ_π and ϕ_y captures the responsiveness of the central bank to current inflation and real activity, respectively, while ρ_i captures the tendency of central banks to only adjust interest rates gradually.[1] Rules of this form are argued to approximate a central bank with a dual mandate to manage both inflation and output fluctuations with additional concerns about the volatility of interest rates.

Less often it is argued Rule (1) shares features with rules that implement an optimal commitment solution. Specifically, Woodford (2003a) and Woodford (2003b) shows that inertial interest rate rules may approximate the necessary history dependence called for by optimal commitment policy. Interest rate smoothing encodes past movements in inflation and output, which shapes expectations of future policy, delivering more favorable trade-offs between inflation and output to policymakers in the face of demand and cost push shocks.

The ability of Rule (1) to approximate optimal policy, however, breaks down when the ZLB is incorporated as a constraint. Most researchers – including us in Figure 1 – capture the ZLB constraint in the following way:

$$i_t = \max\left\{(1-\rho_i)\bar{r} + \rho i_{t-1} + (1-\rho)(\phi_\pi \pi_t + \phi_y y_t), 0\right\}. \tag{2}$$

This rule does not approximate an optimizing central bank confronting the ZLB. The reason is that optimal policy requires policy to be history dependent. At the ZLB, the lagged interest rate no longer capture past economic conditions during the periods in which the ZLB binds. In other words, regardless of what occurs at the ZLB under Rule (2), the monetary policymaker credibly commits to not respond to it. If hyperinflation occurs... so what. Bygones are bygones in the most strict sense in terms of inflation and output realizations when a model is closed with an interest rate rule like (2). Worse yet, the history dependence that is encoded here is the opposite of what an inflation targeting central bank should choose. Upon liftoff, the interest rate smoothing term systematically moves policy away from what a central bank should choose based on current and past inflation and output realizations, which imparts a perverse history dependence to policy.

[1] The discussion here is based on the study of structural monetary models that are solved by standard first-order approximations using piece-wise solutions to capture the non-linearity at the ZLB. This assumption captures nearly the universe of models that are used in central banks and the simple environments in which the forward guidance policy is usually studied. Eggertsson and Singh (2019) shows considering the non-linear environments does not eliminate the puzzles. In other words, the NK puzzles are not an artifact of approximation method.

The exclusion of history dependence in monetary policy at the ZLB when using Rule (2) is all the more stark when juxtaposed to how much history dependence is typically assumed outside of the ZLB and found empirically. When rules like (1) are taken to the data in medium- and large-scale models, the estimates of ρ_i are large. For example, both Smets and Wouters (2007) and Del Negro, Eusepi, Giannoni, Sbordone, Tambalotti, Cocci, Hasegawa and Linder (2013) find ρ_i to be around 0.8. Moreover, when Eggertsson and Woodford (2003) derive optimal policy for a central bank that faces a ZLB constraint, even more history dependence in policy is required!

From the perspective of the agents who inhabit the model, the implied history dependence of Rule (1) and its absence at the ZLB is glaring. To see this, note that equation (1) is equivalent to the following policy rule:

$$i_t = \bar{r} + (1-\rho_i)\sum_{j=0}^{t}\rho_i^j\left(\phi_\pi \pi_{t-j} + \phi_y y_{t-j}\right), \tag{3}$$

Models with either Rule (1) or Rule (3) implement the same equilibrium outcomes in standard linearized economies without a ZLB constraint.[2] Rule (3) though makes clear the degree of history dependence that is implicitly assumed. The central bank responds to geometric averages of past inflation and output with high weight placed on observations that occurred far in the past when ρ_i is large. In other words, the literature has long implicitly assumed, and found robust empirical support for, central banks that are history dependent in their policy setting. Quite naturally, the history that policymakers are responding to is that of past inflation and output, consistent with real world statutory mandates.

The equivalence between (1) and (3) is broken at the ZLB. These two rules do not implement the same equilibrium at the ZLB, or in the medium run after liftoff. To see why, consider a third equivalent representation of monetary policy:

$$i_t = \max\left\{\bar{r} + \phi_\pi \omega_t^\pi + \phi_y \omega_t^y, 0\right\}, \tag{4}$$

[2]The equivalence holds for the rational solution of a first-order approximation of the economy in log-deviation from steady state form when initial conditions are the same, $\bar{r}$ is constant, and there is no monetary policy shock. Monetary policy shocks obviously propagate differently in equilibrium between these two rules. Likewise, if $\bar{r}$ tracks the natural rate shock, then the rules implement different policies, which is discussed in detail in Section 2.

where

$$\begin{aligned} \omega_t^{\pi} &= \omega_{t-1}^{\pi} + (1-\rho_i)(\pi_t - \omega_{t-1}^{\pi}) \\ \omega_t^{y} &= \omega_{t-1}^{y} + (1-\rho_i)(y_t - \omega_{t-1}^{y}) \end{aligned}$$

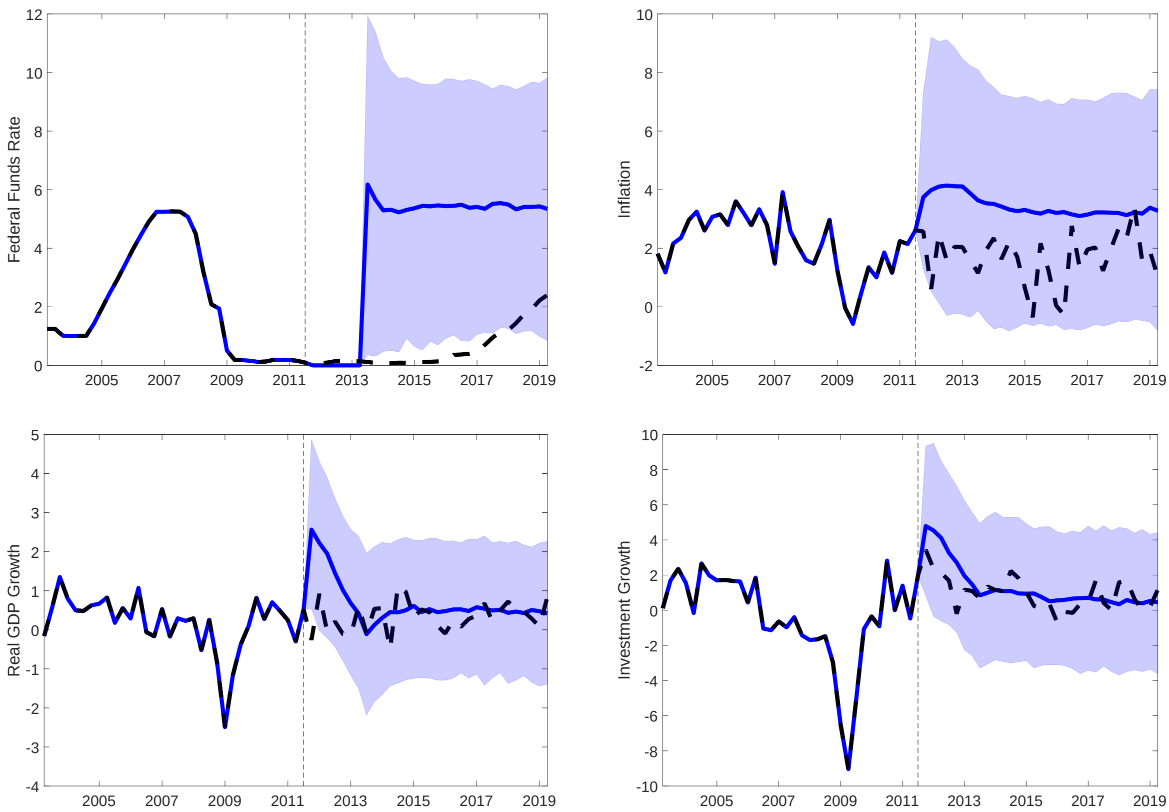
are weighted averages of past inflation and output. Under this description of policy, the missing history dependence assumed at the ZLB under Rule (2) is obvious. Under Rule (2), exogenously setting $i_t = 0$ deletes the central bank and its objectives from the model. In addition, upon lift off of the interest rate, Rule (2) does not return to approximating an optimizing central bank. Instead, the central bank down-weights its response to current output and inflation by doggedly refusing to raise interest rates fast enough. Under Rule (4), however, exogenously setting $i_t = 0$ does not change what the central bank fundamentally cares about in the model. It simply implements exactly what the ZLB represents: a constraint on the choice of the instrument of policy. The central bank and the private sector do not lose their ability to track the evolution of output and inflation during the period interest rates are constrained; nor does the private sector lose their ability to formulate expectations about how policymakers will respond to current misses of their targets in the future.

Put a different way, the ZLB and the standard forward guidance thought experiments in NK models using Rule (2) are better described as what occurs if the central bank credibly abandons inflation targeting both at the ZLB and upon liftoff. Considering the forward guidance puzzle under this alternative framing of policy changes how one views what occurs in Figure 1. If the central bank, when confronted with shocks that take policy to the ZLB, credibly promises to abandon inflation targeting and embarks on a systematic low interest rate policy for the next decade, then a significant and immediate rise in inflation is arguably what most central bankers and economists everywhere would expect *and* is essentially what the NK model predicts. Viewed through this lens, the forward guidance puzzle is perhaps best viewed as a plausible prediction to an extreme policy.

Figure 2 shows how the out-of-sample forecasts change when we transform the interest rate rule of the Smets and Wouters model to the form of Rule (4).[3] We repeat the same forecasting

[3]It is straightforward to verify that it implements the same rational solution when the ZLB is not considered. The exact formulation of the rule we use is given in Section 4.

Figure 2: Calendar-based forward guidance without the forward guidance puzzle



Notes: Out-of-sample forecasts of the August, 9^{th} 2011 Federal Reserve calendar-based forward guidance promise using the model of Smets and Wouters (2007) with the interest rate rule written in the form of (4). All variables show Q/Q forecasts. Inflation and the Federal Funds Rates are annualized.

exercise as in Figure 1. The forecasts no longer explode. The responses are large, but quite sensible. Forward guidance here provides a significant boost to the economy, larger on average than what occurred on average, but not unreasonable given that the policy is assumed perfectly credible in the forecasts. We show that for empirically relevant exercises like this one, the New Keynesian puzzles are more-or-less absent at the posterior estimates of the model parameters under Rule (4). However, for extremely long expected ZLB episodes some puzzles remain. To remove all puzzles, a central bank must follow a rule that more than makes-up for past misses in inflation from the target at the ZLB, which is what Eggertsson and Woodford (2003) show characterizes optimal policy.

The key to understanding the puzzles and their resolutions is history dependence in policy setting. Optimal policy in the NK model requires it. Agents expect policymakers to make-up for misses to their targets during the ZLB episode because the objectives of the central bank – to stabilize inflation and output – do not change because of the ZLB constraint. Realistically, rational agents in the model do not expect that inflation can rise substantially without the central bank raising rates at some point in the future. Importantly, though, rational agents only form these expectations when they have been told explicitly that this is what the central bank will do. Commitment policy featuring the right history dependence does exactly that. In contrast, the typical way the ZLB is implemented as in Rule (2), credibly tells rational agents that the central bank will do something entirely different.

We show this point by revisiting optimal policy in the standard New Keynesian model with two shocks: a natural rate shock and a cost push shock. We show that the unconditional optimal target criteria proposed by Blake (2001) and Jensen and McCallum (2002) may be approximately implemented by either a rule like Rule (1) or Rule (3). By that we mean that either solution implements policy that results in higher welfare than the same non-inertial policy rule. And in the case of cost push shocks, the rules generate qualitatively similar responses in output and inflation when compared to the outcomes under the unconditional optimal target criteria. We then show that this is not true at the ZLB by replicating the thought experiment proposed by Eggertsson and Woodford (2003) of a central bank responding to a real interest rate shock that causes a one-time bind of the ZLB for an unknown duration. Responding to such a shock with the optimal state contingent forward guidance policy and a promise to return to Rule (4) approximates the optimal policy outcomes quite well. In contrast, if the central bank promises to return to Rule (2) following the forward guidance policy, the usual forward guidance puzzle results are observed.

To quantify why these results occur, we derive closed-formed solutions to the standard NK model under a general policy rule that responds to a weighted average of all past inflation. Using the NK puzzle definitions based on Diba and Loisel (2021), we show that policy must be more history dependent than a price level targeting regime to eliminate the puzzles. Policy must follow a rule that more than makes up for past misses at the ZLB. Although smaller amounts of history dependence, e.g., price level targeting or responding to a weighted average of inflation, can ameliorate the forward guidance and fiscal multiplier puzzles to a substantial degree and may also eliminate the paradox of flexibility and toil in some cases.

We believe there are three important takeaways from this result. First, the representative agent NK (RANK) model has fewer flaws when quantitatively capturing ZLB episodes than is currently recognized. Second, even without the NK puzzles, the RANK model predicts incredibly powerful general equilibrium effects. For example, the optimal policy recommendation of

Eggertsson and Woodford (2003), which show that a central bank can almost perfectly stabilize the economy at the ZLB for an arbitrarily large shock, does not rely on, or is an example of, the forward guidance puzzle. Therefore, the large literature that has developed to eliminate the puzzles by dampening general equilibrium effects in the model, which the authors of this paper have contributed to, still has an important role to play in explaining economic dynamics and in the design of policy. Finally, our results echo the concerns raised by Brassil, Ryan and Yadav (2023) on the misuse of ad hoc policy rules and argue for caution when modifying interest rate rules in isolation. It has become standard to ignore optimizing behavior by policymakers in quantitative structural modeling with researcher studying a variety of modifications to policy rules and describing them as capturing real world policy objectives when in fact there is no such connection justified in the model.

1.1 Related Literature

The New Keynesian puzzles literature is large and has two main branches. The most significant branch from a quantitative and policy perspective is on the forward guidance puzzle. The seminal papers are Del Negro et al. (2012) and Carlstrom et al. (2015), which both illustrate the problem – explosive responses of output and inflation to promised pegs of the interest rate – and posit solutions. The latter show that sticky information can reduce or eliminate puzzles while the former argues that the credibility of the policy at longer horizons is implausible. In fact, Del Negro et al. (2012)'s point on credibility of the policy makes a similar argument to ours. The implications of the forward guidance thought experiment imply future movements of interest rates that are implausible. We quantify their insight and connect it to optimal policy.

Many papers have sought to eliminate the forward guidance puzzle by changing primitive assumptions of the NK model. For example, Del Negro, Giannoni and Patterson (2023) makes use of a perpetual youth model to micro found a reduction in the horizon of agents' expectations delivering the same ameliorative effect on forward guidance as imperfect credibility. Further refinements on using credibility to resolve the puzzle are studied by Haberis, Harrison and Waldron (2019) and Gibbs and McClung (2023). They both show that partial credibility of holding rates at zero resolves the forward guidance puzzle. Andrade, Gaballo, Mengus and Mojon (2019) study the case where some agents interpret forward guidance as worse economic conditions rather than a credible promise of stimulus, which also ameliorates the effect of such policies.

There are many studies on information frictions or bounded rationality as a way of resolving the puzzle such as Kiley (2016) using sticky information; Angeletos and Lian (2018), Farhi and Werning (2019), and Evans, Gibbs and McGough (2022) using level-k reasoning; Gabaix (2020) myopia; and Eusepi, Gibbs and Preston (2022) who use adaptive learning. Each these bounded rationality papers makes use of the fact that the significant impacts of forward guidance come through the general equilibrium effects of expectations. Bounded rationality lowers these effects and may eliminate the puzzles.

Another sub-strand of this literature is incomplete markets. McKay, Nakamura and Steinsson (2016) and Hagedorn, Luo, Manovskii and Mitman (2019) show that incomplete markets in a heterogeneous agent NK model can resolve the puzzle. Eggertsson, Mehrotra and Robbins (2019) show that the puzzle is absent in overlapping generation models with debt constraints. However, Farhi and Werning (2019) and Bilbiie (2020) show that incomplete market is not a robust way to eliminate the forward guidance puzzle.

In addition to these behavioral approaches, Cochrane (2017, 2023), McClung (2021), Gibbs and McClung (2023), and Diba and Loisel (2021) show that alternative monetary and fiscal policy frameworks also can eliminate the puzzles. The last paper listed shows that when monetary policy is conducted via money supply rules that the economy is puzzle free under an interest rate peg. The remaining papers show that closing a model with active fiscal policy as under the Fiscal Theory of the Price Level provides a puzzle free equilibrium in the NK model. Our paper fits into this strand of the literature. The policies studied by these authors resolve the forward guidance puzzle because the relevant history dependence of policy is maintained during the period of constrained interest rates.

The second smaller branch of this literature focuses on the other puzzles: fiscal multipliers, toil, and flexibility. The fiscal multiplier puzzle is essentially the same as the forward guidance puzzle in that anticipated fiscal policy has implausibly large effects. Hills and Nakata (2018) offer a resolution for this puzzle that involves tracking a shadow rate at the ZLB, which encodes the correct history dependence. Eggertsson (2010) identified the paradox of toil as the result that negative supply shocks are expansionary at the ZLB. The paradox of flexibility is defined in Eggertsson and Krugman (2012) and is the result that reductions in price stickiness increases the relative strength of all the other puzzles at the ZLB, i.e., in the face of current and anticipated shocks, the ZLB constraint is even more disruptive in the model as price flexibility increases. These puzzles are often, but not always, subordinate to the forward guidance puzzle. If you eliminate the forward guidance puzzle, then typically that removes the other puzzles as well but not vice versa. Wieland (2019) is notable here for providing an empirical test for the paradox of toil. No empirical support is found for this puzzle, which is consistent with our argument that monetary policy in reality is not described well by Rule (2).

2 Inertial Interest Rate Rules and approximating optimal policy

In this section, we revisit optimal policy in the standard New Keynesian environment under the timeless perspective. We show that optimal commitment policy in the NK model may be approximated in the absence of the ZLB by either an interest rate rule like (1), the standard inertial rule, or by a rule like (4), which responds to weighted averages of past inflation and output, which we call a weighted average rule. The approximations, however, diverge greatly at the ZLB.

2.1 Classic optimal policy in the NK model and inertial rules

Consider the standard optimal monetary policy problem with commitment from the timeless perspective (i.e., where the central bank does not seek to exploit initial condition) described by Clarida, Gali and Gertler (1999) or Woodford (2003a). The central bank seeks to maximize household welfare by committing to a policy now and into the infinite future to negate the effects of stochastic shocks to the real interest rate and to markups. To achieve this goal, the central bank seeks to minimize

$$\min_{\pi_t, y_t, i_t} \left\{ \frac{1}{2} E_t \sum_{T=t}^{\infty} \beta^{T-t} (\pi_t^2 + \alpha y_t^2) \right\}, \tag{5}$$

which is a quadratic approximation to household utility where α is the weight given to variation in output (y_t) relative to inflation (π_t). Alternatively, we may view the loss function as that of an inflation targeting central bank with a dual mandate. The central bank seeks to minimize deviation in inflation from a target ($\bar{\pi} = 0$ in this case) while considering the output costs when implementing policy. We assert that most modelers roughly have in mind an objective of the form of (5) whenever they write down a reduced form monetary policy rule. The tacit assumption in the literature is that Taylor-type monetary policy rules approximately implement policy that minimizes (5).

That assumption is mostly true. Woodford (2001) shows this for non-inertial interest rate rules and Woodford (2003b) shows it is true of inertial interest rate rules that respond to lagged interest rates. However, its not true a the ZLB. To understand why, consider the optimal policy problem of a central bank that seeks to minimize (5) taking as given the first order conditions for household's and firm's decisions:

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}\left(i_t - E_t\pi_{t+1} - (\bar{r} - r_t^n)\right) \tag{6}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t + \mu_t \tag{7}$$

where Equations (6) and (7) are the standard NK IS and Phillips curves log-linearized around a zero inflation steady state, i_t is the policy rate, σ is the inverse intertemporal elasticity of substitution, r_t^n is a stochastic real interest rate shock, β measures the rate of time preference, κ is composite parameter capturing the degree of price rigidity, and μ_t is a stochastic cost push shock. When the ZLB is considered as a possible constraint, the solution must also satisfy $i_t \geq 0$.

2.2 Approximating policy without the ZLB

The optimal target criterion that minimizes the central bank's loss function (5) from the timeless perspective, ignoring the ZLB constraint, is

$$y_t - y_{t-1} = -\frac{\kappa}{\alpha}\pi_t. \tag{8}$$

However, for our purposes it is more convenient to study a slight refinement to the target criteria: the *unconditional* optimal target criteria proposed by Jensen and McCallum (2002) and Blake (2001). The former shows numerically and the latter analytically that welfare on average is improved when time discounting by the central bank is ignored resulting in the targeting criteria:

$$y_t - \beta y_{t-1} = -\frac{\kappa}{\alpha}\pi_t. \tag{9}$$

This criterion is convenient because we can write this as

$$(1-\beta L)y_t = -\frac{\kappa}{\alpha}\pi_t$$

where L is the lag operator. The inverse of $(1-\beta L)$ exists provided $|\beta| < 1$. Therefore, we can write

$$y_t = -\frac{\kappa}{\alpha}\frac{\pi_t}{1-\beta L}.$$

Proposition 1: *The optimal target criterion (9) may be implemented by either of the following interest rate rules*

$$\text{Optimal Rule 1} \quad \begin{cases} i_t &= \frac{\sigma\kappa}{\alpha(1-\beta)}\omega_t^\pi + \sigma E_t y_{t+1} + E_t\pi_{t+1} + r_t^n \\ \omega_t^\pi &= \omega_{t-1}^\pi + (1-\beta)(\pi_t - \omega_{t-1}^\pi) \end{cases} \tag{10}$$

$$\text{Optimal Rule 2} \quad i_t = \beta i_{t-1} + \frac{\sigma\kappa}{\alpha}\pi_t + (1-\beta L)\left(\sigma E_t y_{t+1} + E_t\pi_{t+1} + r_t^n\right) \tag{11}$$

Therefore, we can implement optimal policy with an interest rate rule that responds to a weighted average of inflation like Rule (3) or we can implement the same policy with a rule that responds to lagged interest rates like Rule (1). Either formulation implements the same equilibrium outcomes when the ZLB is not a constraint on policy.

Figure 3 shows how well Rule (1),

$$i_t = (1-\rho_i)\bar{r} + \rho_i i_{t-1} + (1-\rho_i)(\phi_\pi\pi_t + \phi_y y_t),$$

and Rule (3),

$$\begin{aligned} i_t &= \bar{r} + \phi_\pi\omega_t^\pi + \phi_y\omega_t^y \\ \omega_t^z &= \omega_{t-1}^z + (1-\rho_i)(\pi_t - \omega_{t-1}^z) \text{ for } z = \pi \text{ or } y, \end{aligned}$$

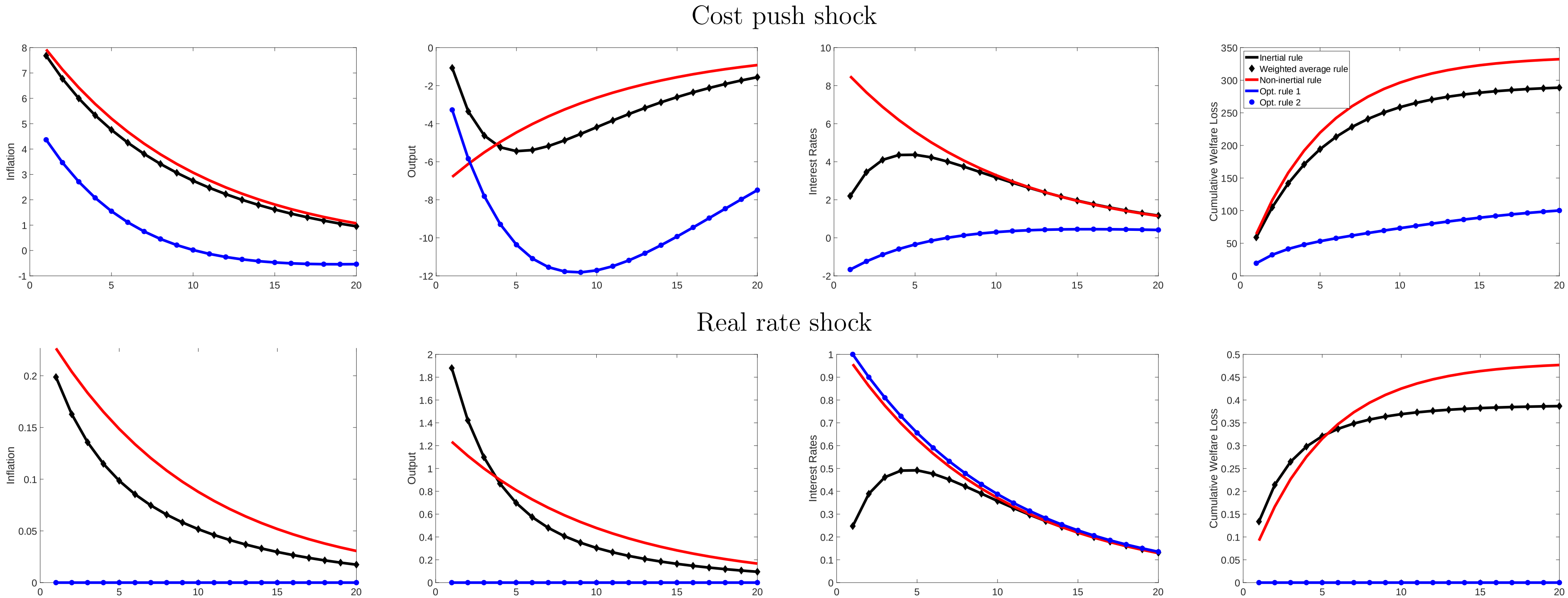
approximate optimal policy in response to the two shocks in comparison to a non-inertial Taylor-type rule. We set $\rho_i = 0.8$, $\phi_\pi = 1.5$, and $\phi_y = 0.5$ for the inertial/weighted average interest rate rules and use the same parameters except with $\rho_i = 0$ for the non-inertial rule. We set $\alpha = \kappa/(\sigma\phi_\pi)$ for the optimal policy rules so that the weight placed on stabilizing inflation relative to output is comparable to the standard interest rate rule coefficients.[4] We assume the

[4]The remaining parameters in the simulation follow Eggertsson and Woodford (2003) and are set to $\beta = 0.99$, $\sigma = 0.5$, and $\kappa = 0.02$. The same conclusions holds between the standard rules and optimal policy if we set

shocks follow AR(1) process with persistence of 0.9. The top set of figures shows the response to the cost push shock while bottom figures show the real interest rate shock. The right panels of the figure shows the cumulative welfare loss.

$\phi_\pi = \kappa/(\sigma\alpha)$ and α at its welfare theoretic value. The welfare theoretic value of α is $\alpha = \theta/\kappa$ with $\theta = 7.87$.

Figure 3: Standard rules versus optimal rules without the ZLB



Notes: Comparing impulse responses for different policy rules to a cost push (top row) and a real rate (bottow row shock. The inertial rule is rule (1) with $\rho_i = 0.8$, $\phi_\pi = 0.1.5$, $\phi_y = 0.5$. The weighted average rule is (3) with the same parameters as the inertial rule. The non-inertial rule is the same as (1) with $\rho_i = 0$. The optimal rules are (22) and (23), respectively.

The inertial and weighted average rules approximate optimal policy in two ways. First, they generates lower cumulative loss in response to the shocks than the non-inertial rule. Second, they generates output dynamics that are qualitatively like those induced by optimal policy in response to the cost push shock. Therefore, in a sense, standard Taylor-type inertial rule does approximately implement policy consistent with the narrative most researchers construct around the NK model.

Surprisingly, though, the two rule's approximation to optimal policy is worse for the real interest shock than the cost push shock. A defining feature of the NK model is the divine coincidence, which is the result that policymakers can perfectly offset demand shocks in the simple model (see Blanchard and Galí, 2007) and which is clearly seen in Figure 3 for the optimal rules. This fact foreshadows why inertial interest rate rules suffer at the ZLB. The ZLB in effect is a demand shock. Weighted average rule allows the central bank to adjust the interest rate one-for-one with a real interest shock without further consequence for policy setting. This is not possible with the inertial rule. Responding one-for-one to a real interest rate movement mechanically induces futures changes in the policy rate even though none are required.

In other words, lagged interest encode the wrong history dependence in response to demand shocks. We can see this by comparing how r_t^n enters the optimal rules in Proposition 1. Rule (11) with a lag of the nominal interest rate requires lags of expectations and the shock to correct for the non-optimal history dependence embodied by i_{t-1}. The weighted average rule (10) does not have this problem. We can simply replace $\bar{r}$ in the weighted average rule with r_t^n and the rule can implement the divine coincidence without further effects on policy setting.

2.3 Approximating policy with the ZLB

The importance of history dependence only grows at the ZLB. Eggertsson and Woodford (2003) show that the optimal target criterion when $i_t \geq 0$ is a time-varying price level target:

$$\begin{aligned} \tilde{p}_t &= p_t^* \\ \tilde{p}_t &= p_t + \frac{\alpha}{\kappa} x_t \\ p_{t+1}^* &= p_t^* + \beta^{-1}\left((1+\kappa\sigma^{-1}) - L\right)(p_t^* - \tilde{p}_t), \end{aligned}$$

where $\tilde{p}_t$ adjusts according to past misses in that target $(p^*_t - \tilde{p}_t)$ and L is again the lag operator. This criterion requires the central bank to be extremely history dependent. The central bank should not just be a price level targeter – perfectly make-up for past misses. It should promise to permanently overshoot on the price level in response to shocks that cause the ZLB to bind – more than make-up for past misses.

Implementing or approximating the optimal target criterion in this case using an interest rate rule requires forward guidance. A central bank that follows Rule (2) or Rule (4) may announce how long they intend to hold interest rates at zero in response to a shock that causes them to optimally set interest rate at zero in the current period. That promise is paired with a promise to return to same interest rule following the conclusion of the zero-rate policy.

How closely do Rule (2) and Rule (4) implement the optimal outcomes for the optimal forward guidance policy? To answer this question, we proceed in two steps. First, we replicate the ZLB thought experiment explored by Eggertsson and Woodford (2003) of optimally responding to a real interest rate shock of uncertain duration that causes the ZLB to bind. The optimal policy to this shock implies a state-contingent forward guidance promise, where the number of quarters of zero interest policy is indexed by the expected duration of the shock. Second, we use the optimal state-contingent forward guidance promise to respond to the same shock but assume that policy returns to either Rule (2) or Rule (4) rather than optimal policy. We compare the equilibrium outcomes.

The real interest rate shock follows a two-state reducible Markov process. In period one, the shock occurs and $r^n_t = r_S < 0$. We call this state S. The shock remains in effect in each period with probability $(1 - \delta)$. With complementary probability δ, the real interest rate returns to steady state, $r^n_t = r_N = \bar{r} > 0$. We call this state N. The central bank and the private sector understand the shock process. The central bank responds in the period the shock occurs with a state-contingent forward guidance promise, where for every possible duration of the shock, $\tau = 1, 2, 3, ...$, the central bank provides a promised duration of additional periods of zero interest rate policy in state N, e.g., $k_\tau = \{1, 2, 2, 3, 3, 4, ...\}$, which generate the appropriate initial conditions for optimal policy forever after.

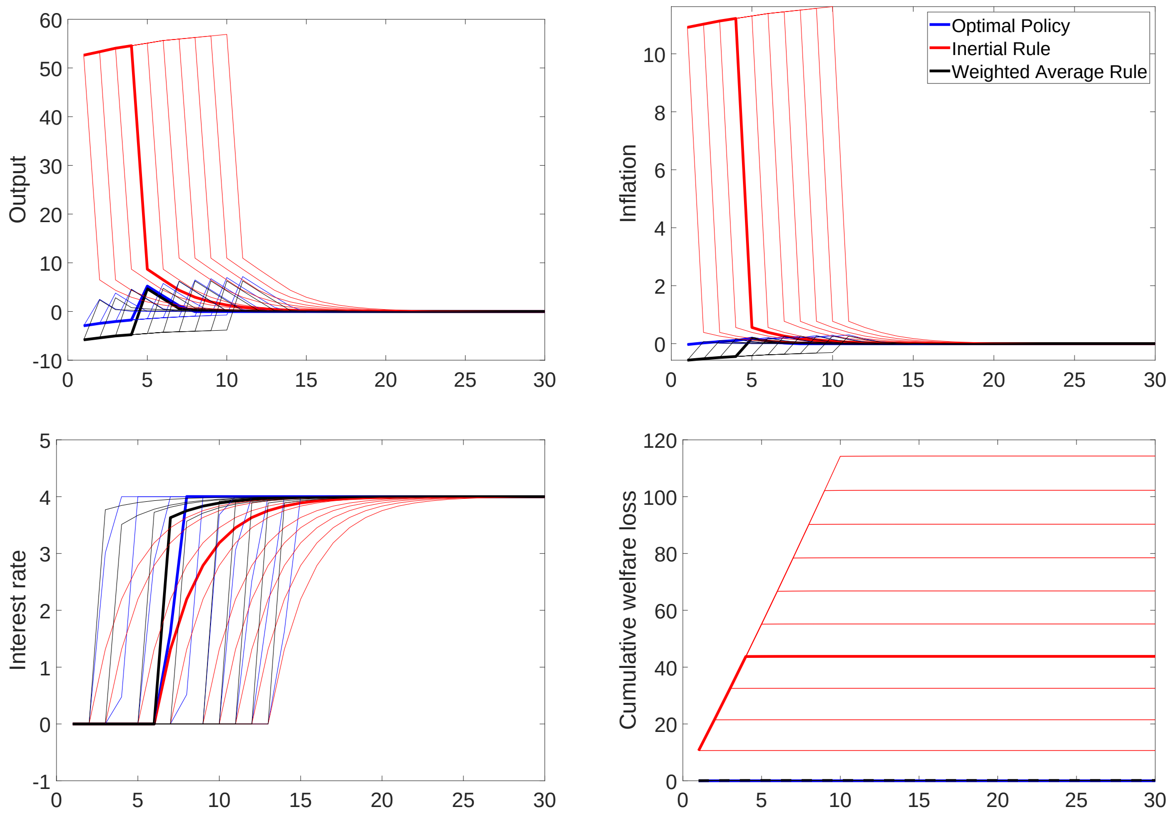
Figure 4 shows the equilibrium outcomes under optimal policy compared to the approximations using either Rule (2) or Rule (4) for realizations of the shocks lasting one through ten quarters. For comparison purposes, we have highlighted the paths associated with a shock lasting four quarters. We use the same parameters for the model as in Figure 3 and follow Eggertsson and Woodford (2003) with $r_N = 0.1$, $r_S = -0.005$ and $\delta = 0.1$.[5] By construction, the forward guidance policy is the same under all three specifications. The only difference is the policy pursued after the interest rate lifts off from zero. The equivalence of Rule (2) and Rule (4) is broken. The two rules generate extremely different equilibrium dynamics and welfare outcomes in response to the same shock and forward guidance policy. The weighted average rule continues to approximate optimal policy, while the inertial rule exhibits the forward guidance puzzle.

To understand the economics of the different outcomes, compare the highlighted paths of the interest rate for a shock that lasts four quarters in the to bottom left panel of Figure 4. Optimal policy calls for a return to the neutral rate two quarters after liftoff. The weighted average rule returns policy to neutral in about eight quarters after liftoff. The inertial rule, however, does not return policy to neutral for more than 24 quarters or six years after liftoff. Moreover, this policy is pursued despite the fact it is known with certainty that no further shocks will occur. Policymakers are promising to systematically make errors for years because policy is dependent on past interest rate realizations rather than past inflation and output realizations. Policymakers here have explicitly abandoned their dual mandate.

Figure 5 further illustrates the role of that history dependence plays. Here we plot the time zero expected paths for output under the inertial and weighted average rules for different values for ρ_i. Specifically, we simulate the model for a large number of realizations of the shock and then weight the individual outcomes by the probability with which they occur to summarize the expected outcomes of the different policies. The solid blue line shows the outcomes under optimal policy. The closest line to it is the dashed black line that corresponds to the weighted average rule with $\rho_i = 0.95$. As we decrease ρ_i, the approximation to optimal policy gets worse. The opposite relationship occurs for the inertial rule. Less history dependence - ρ_i closer to zero - generates outcomes closer to optimal policy. When $\rho_i = 0$, the weighted average rule and the inertial rule are the same and equal a non-inertial rule. Therefore, inertial rules encode the wrong history dependence at the ZLB.

[5] We follow the solution method described by Eusepi et al. (2022), which is found in their online appendix. The method is based on the original solution algorithm proposed by Eggertsson and Woodford (2003) and described more recently in Eggertsson, Egiev, Lin, Platzer and Riva (2021).

Figure 4: Standard rules versus optimal rules without the ZLB



Notes: Each line corresponds to a different realization of the Markov shock process. The outcomes for a shock lasting four quarters are highlighted to make comparisons across specifications easier. Inflation and interest rates are expressed in annual terms.

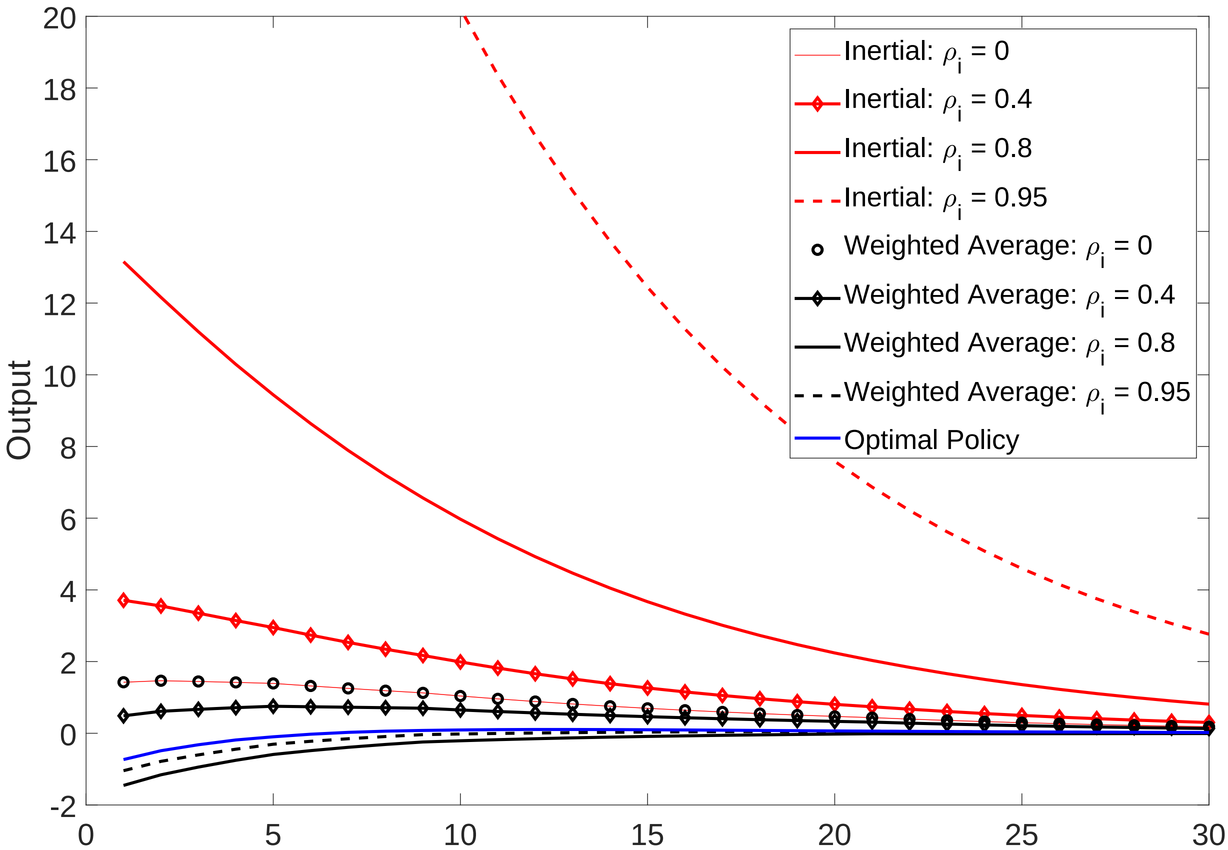
A further implication of Figure 5 is that the stabilization outcomes that are possible under optimal policy are not due to the forward guidance puzzle. Gibbs and McClung (2023) provide a sufficient condition to rule out the forward guidance puzzle as present and it is straightforward to numerically verify that optimal policy satisfies it under standard calibrations. Why this is true is made clear in the next section.

3 Resolving New Keynesian Puzzle

We now turn to quantifying how much of the *right* history dependence is required to eliminate the NK puzzles completely. We do so by generalizing the definitions of the NK puzzle put forward by Diba and Loisel (2021) and analyzing the consequence of following a weighted average rule with different values for ρ_i.

To accommodate Diba and Loisel (2021) definitions, we modify the NK model defined by equations (6) and (7) in several ways. First, we remove the cost push shock from the model and

Figure 5: Approximating optimal policy as a function of history dependence



Notes: Each line corresponds to the time zero expectation of the path of output. We simulate the model for a large number of realizations of the shock and then weight the individual outcomes by the probability with which they occur.

replace it with a supply shock that represents variation in marginal cost from changes to labor supply (a_t). This shock allows us to explore the paradox of toil. Second, we add a government spending shock that may affect both supply and demand in the economy (g_t). This shock allows us to explore the fiscal multiplier puzzle. Finally, we assume that all shocks are i.i.d. without exogenous persistence. The NK model of interest is given by

$$\begin{aligned} y_t &= E_t y_{t+1} - \sigma^{-1}\left(i_t - E_t\pi_{t+1} - r_t^n\right) + g_t - E_t g_{t+1} && (12) \\ \pi_t &= \beta E_t \pi_{t+1} + \kappa\left(y_t - \delta_g g_t - a_t\right). && (13) \end{aligned}$$

The NK puzzles are defined by way of a thought of experiment. Contemplate the effect of an anticipated shock that is known at time $t = T$ but occurs in time $t = T^* > T$ under the following monetary policy regime:

$$i_t = \begin{cases} \bar{i} + \phi\pi_t & \text{for } t = T, T+1, ..., T^* \\ \bar{i} + \phi^*\pi_t & \text{for } t > T^*, \end{cases} \tag{14}$$

where $0 \leq \phi < 1$ and $\phi^* > 1$. Then, ask what happens to the effect of the shock in time T if the same shock occurs at even later date, i.e., as $\Delta_p = T^* - T > 0$ goes infinity.

We can derive a closed-form solution to the equilibrium effect of this model by using the Phillips curve to eliminate y_t in the IS curve. We can express the model for $t < T^*$ as

$$\left(\beta L^{-2} - (\beta + 1 + \frac{\kappa}{\sigma})L^{-1} + \left(1 + \frac{\kappa\phi}{\sigma}\right)\right)\pi_t = X_t$$

$$X_t \equiv -\frac{\kappa}{\sigma}(\bar{i} - r_t^n) - \kappa(a_t - E_t a_{t+1}) + \kappa(1-\delta_g)\left(g_t - E_t g_{t+1}\right), \tag{15}$$

where L is the lag operator. Factoring the lag polynomial we have

$$(L^{-1} - \lambda_1)(L^{-1} - \lambda_2)\pi_t = X_t$$

where the eigenvalues for the economically relevant parameters satisfy $0 \leq \lambda_1 < 1 < \lambda_2$. Using the method of partial fractions, we can write this as

$$\pi_t = \frac{1}{\lambda_2 - \lambda_1} E_t\left[\frac{\lambda_1^{-1}}{1 - (\lambda_1 L)^{-1}} - \frac{\lambda_2^{-1}}{1 - (\lambda_2 L)^{-1}}\right] X_t.$$

Finally, under the assumed monetary policy and shocks processes it follows that $E_t\pi_{T+T^*+j} = 0$ for all $j > 0$, which provide the necessary limit conditions to construct an unique rational expectation equilibrium by solving the model forward in time:

$$
\begin{aligned}
\pi_t &= \frac{1}{\lambda_2 - \lambda_1} E_t \left[\sum_{T=t}^{T^*} \left(\left(\frac{1}{\lambda_1}\right)^{T-t+1} - \left(\frac{1}{\lambda_2}\right)^{T-t+1} \right) X_T \right] \\
y_t &= \frac{1}{\kappa(\lambda_2 - \lambda_1)} E_t \left[\sum_{T=t+1}^{T^*} \left((1-\beta\lambda_1)\left(\frac{1}{\lambda_1}\right)^{T-t+1} - (1-\beta\lambda_2)\left(\frac{1}{\lambda_2}\right)^{T-t+1} \right) X_T \right] \\
&+ \frac{\lambda_1^{-1} - \lambda_2^{-1}}{\kappa\left(\lambda_2 - \lambda_1\right)} X_t + \delta_g g_t + a_t
\end{aligned}
$$

Diba and Loisel (2021) define The NK puzzles by studying the properties of dynamic multipliers for a shock as $\Delta_p \to \infty$,

$$\lim_{\Delta_p \to +\infty} \partial z_t / \partial X_{m,t+\Delta_p} = \infty \text{ where } z \in \{\pi, x\} \text{ and } X_m \in \{i^*, g, a\}$$

Definition 1 (forward guidance puzzle) *When the policy rate is expected to be set passively during the next $\Delta_p > 0$ periods, the response of current inflation and output to an expected policy-rate shock $i^* \neq \bar{i}$, Δ_p periods ahead, goes to infinity with Δ_p i.e.*

$$\lim_{\Delta_p \to +\infty} \partial z_t / \partial i_{t+\Delta_p} = \infty \textit{ where } z \in \{\pi, x\}$$

.

Definition 2 (fiscal multiplier puzzle) *When the policy rate is expected to be set passively during the next $\Delta_p > 0$ periods, the response of current inflation and output to an expected expansionary government spending shock, $g_t > 0$, Δ_p periods ahead, goes to positive infinity with Δ_p, i.e.,*

$$\lim_{\Delta_p \to +\infty} \partial z_t / \partial g_{t+\Delta_p} = \infty \textit{ where } z \in \{\pi, x\}$$

.

Definition 3 (paradox of toil) *When the policy rate is expected to be set passively during the next $\Delta_p > 0$ periods, the response of current output to a positive supply shock, $a_t > 0$, Δ_p*

periods ahead, is weakly contractionary with Δ_p, i.e.,

$$\lim_{\Delta_p \to +\infty} \partial z_t / \partial a_{t+\Delta_p} \leq 0 \text{ where } z \in \{\pi, x\}$$

.

Definition 4 (paradox of flexibility) *When the policy rate is expected to be set passively during the next $\Delta_p > 0$ periods, the response of current of inflation and output to an expected shock Δ_p periods ahead goes to positive or negative infinity as κ goes to infinity, i.e.,*

$$\lim_{\kappa \to +\infty} \partial z_t / \partial v_{t+\Delta_p} = \pm\infty \text{ where } z \in \{\pi, x\} \text{ and } v = \{i^*, r^n, g, a\}$$

.

Three of the four puzzles are direct consequence of the properties of λ_1. The forward guidance puzzle and the fiscal multiplier puzzles are caused by the fact that $\lambda_1 < 1$, which makes $\lambda_1^{-\Delta_p}$ grow without bound as Δ_p increases. The paradox of flexibility is due to the fact that $\lambda_1 \to 0$ as $\kappa \to \infty$. The remaining puzzle occurs because the properties of X_t with a negative sign always appearing in front of the anticipated supply shocks in equilibrium.

The definition of the forward guidance puzzle we study here is a looser definition than is sometimes applied in the literature. It requires that response does not grow without bound when Δ_p is increased. However, Farhi and Werning (2019) argue that another way to think of this puzzle is in terms of what they call an anti-horizon effect. Because the agents discount the future, it is counter intuitive that an identical policy would have a larger effect when its implementation is moved farther into the future all else equal. By this definition, the forward guidance puzzle is present even if the response is bounded so long as it is increasing with Δ_p. Eliminating the puzzle requires the response of output and inflation to be a decreasing function of Δ_p. We consider this stronger definition as well.

3.1 The Puzzles under weighted average policy rule

Consider the NK puzzle thought experiment under the following monetary policy regime

$$i_t = \begin{cases} \bar{i} + \phi\pi_t & \text{for } t = T, T+1, ..., T^* \\ \bar{i} + \phi^*\omega_t & \text{for } t > T^*, \end{cases} \tag{16}$$

$$\omega_t^\pi = \begin{cases} \rho\omega_{t-1} + \pi_t & \text{for } t = T, T+1, ..., T^* \\ \rho^*\omega_{t-1} + \pi_t & \text{for } t > T^*, \end{cases} \tag{17}$$

where the interest rate responds only to current inflation for some period but it is expected to respond to a weighted average of inflation in the future. The idea is that agents understand that past inflation outcomes matter for policy in the future even if those past outcomes do not currently matter for the setting of the policy rate. This is the situation that is typically modeled at the ZLB. We can nest that situation explicitly by setting $\bar{i} = -\bar{r}$ and $\phi = 0$. We can also approximate more-than-make-up policy required by optimal commitment at the ZLB by setting $\rho > 1$.

The question of interest is how large must ρ be in order to remove the puzzle. The non-linearity we introduce here makes a tractable analysis of the full model difficult. However, we can provide analytic quantification by studying two special cases that illustrates how history dependence solves the puzzles. It is straightforward to numerically verify that the results hold in the full model and generalize to larger scale models, which we show in Section 4.

3.2 Simple case when $\beta = 0$

We begin with a simplified version of the NK model where we set $\beta = 0$. The economy is assumed to be at steady state in time $t = T - 1$ such that $\pi_{T-1} = \omega_{T-1} = 0$. In time $t = T$, a shock is anticipated to occur in period $T^* > T$. This shock is the only shock and thus we study the perfect foresight solution. When $t \geq T^* + 1$, the economy evolves as

$$\begin{aligned} \pi_t &= \frac{(\kappa + \sigma)}{\sigma}\pi_{t+1} - \frac{\kappa\phi^*}{\sigma}\omega_t + X_t \\ \omega_t &= \rho^*\omega_{t-1} + \pi_t \end{aligned}$$

where X_t is defined the same as in equation (15). We call this the terminal regime. When $T < t \leq T^*$, the economy evolves according

$$\begin{aligned} \psi \pi_t &= \pi_{t+1} + \frac{1}{1+\kappa\sigma^{-1}} X_t \\ \omega_t &= \rho \omega_{t-1} + \pi_t \end{aligned}$$

where $\psi = \frac{1+\kappa\sigma^{-1}\phi}{1+\kappa\sigma^{-1}} \in [0,1)$ and ω_t is decoupled from inflation except to record it history during the passive policy period.

In the terminal regime, the minimum state variable solution for inflation is given by

$$\pi_t = -\Omega^* \omega_{t-1} + C^* X_t$$

where $\Omega^* > 0$ and C^* are functions of ϕ^* and ρ^* and where for convenience we factor out a negative one from Ω^*.[6] The perfect foresight expectation of inflation for time $T^* + 1$ formed in time T is given by

$$E_T \pi_{T^*+1} = -E_T \Omega^* \omega_{T^*} = -E_T \Omega^* \sum_{j=0}^{\Delta_p} \rho^j \pi_{T^*-j},$$

where the last equality holds because in time T^* we are in the passive regime. Using this expectation, we solve for the perfect foresight path of inflation given the shock and policy. Starting with time $t = T^*$,

$$\psi \pi_{T^*} = -\Omega^* \sum_{j=0}^{\Delta_p} \rho^j \pi_{T^*-j} + \frac{1}{1+\kappa\sigma^{-1}} X_{T^*}.$$

Solving for π_{T^*}, we have

$$\pi_{T^*} = -(\psi + \Omega^*)^{-1} \left(\Omega^* \sum_{j=1}^{\Delta_p} \rho^j \pi_{T^*-j} - \frac{1}{1+\kappa\sigma^{-1}} X_{T^*} \right).$$

[6] We provide the closed-form solution for Ω^* and C^* in the appendix.

Repeating the process for π_{T^*-1}, we have

$$\psi\pi_{T^*-1} = -(\psi+\Omega^*)^{-1}\left(\Omega^*\sum_{j=1}^{\Delta_p}\rho^j\pi_{T^*-j} - \frac{1}{1+\kappa\sigma^{-1}}X_{T^*}\right) + \frac{1}{1+\kappa\sigma^{-1}}X_{T^*-1}.$$

Solving for π_{T^*-1}, we have

$$\pi_{T^*-1} = -(\psi(\psi+\Omega^*)+\Omega^*\rho)^{-1}\left(\Omega^*\sum_{j=2}^{\Delta_p}\rho^j\pi_{T^*-j} - \frac{X_{T^*}+(\psi+\Omega^*)X_{T^*-1}}{1+\kappa\sigma^{-1}}\right),$$

where X_t shows up a second time because of the expectations of the shock within the definition of X_t. Working backwards in this way until time T yields

$$\pi_T = \Phi(\Delta_p)^{-1}\frac{X_{T^*}+(\psi+\Omega^*)X_{T^*-1}}{1+\kappa\sigma^{-1}} \tag{18}$$

where

$$\Phi(K) = \psi^{K+1} + \Omega^*(\psi^K + \rho\psi^{K-1} + \rho^2\psi^{K-2} + \ldots + \rho^{K-1}\psi + \rho^K). \tag{19}$$

The dynamic multiplier of interest for any shock X_m is therefore

$$\frac{\partial\pi_T}{\partial X_{m,t+\Delta_p}} = \frac{C_m}{1+\kappa\sigma^{-1}}\Phi(\Delta_p)^{-1},$$

where C_m is the appropriate constant from the mth shock.

Proposition 2: *The NK model (12), (13), (16), and (17) with $\beta = 0$, $\phi^* > 1$, $0 \le \phi < 1$, and $0 < \rho^* < 1$ has the following properties*

1. *The equilibrium does not exhibit the forward guidance puzzle, fiscal multiplier puzzle, or paradox of flexibility if $\rho > 1$.*

2. *The equilibrium exhibits the forward guidance and fiscal multiplier puzzles for $0 \le \rho < 1$, and $\rho^* \ne \bar{\rho}$ where $\bar{\rho} \in [0, 1)$ is defined in the appendix. If $\rho^* = \bar{\rho}$ then the equilibrium exhibits the forward guidance puzzle, but does not exhibit the fiscal multiplier puzzle.*

3. *The magnitude of the inflation/output response to anticipated monetary or fiscal policy shocks is decreasing in ρ.*

4. *The equilibrium exhibits the paradox of toil if and if only ρ^* is sufficiently small such that $\rho^* \leq \bar{\rho}$ where $\bar{\rho} \in [0, 1)$.*

5. *The equilibrium exhibits the paradox of flexibility if and only if $\rho = 0$.*

The limit of $\Phi(K)$ as $K \to \infty$ is either 0 or ∞ depending on the values of ρ. If $0 \leq \rho < 1$, then $\Phi(K)$ goes to zero. If $\rho > 1$, then the limit is infinity. And if $\rho = 1$, then the limit is $(1-\psi)^{-1}\Omega^*$. Therefore, the forward guidance and fiscal multiplier puzzles cannot be completely eliminated by a simple weighted average rule with geometric decay. It requires make-up policy as prescribed by optimal commitment policy. Importantly, though, the size of the multiplier is decreasing in ρ, which is why the forward guidance puzzle is so diminished in our quantitative example in the Introduction when ρ was large but less than one.

The paradox of toil result depends on the terminal solution Ω^*. The current impact inherits the anticipated effect of Ω^* from when the ZLB no longer binds. Therefore, it implies the *correct* expected sign for any shock. Because of the fact that monetary policy is history dependent, the correct sign of the shock propagates backwards. Therefore, if ρ^* is large enough, the sign of the multiplier on the supply shock flips to the correct sign. Interestingly, the same threshold for ρ^* matters for the qualitative effects of the fiscal shock. If $\rho^* < \bar{\rho}$ ($\rho^* > \bar{\rho}$) then the anticipated monetary policy shock raises (lowers) output. However, if $\rho = \bar{\rho}$ exactly then the government spending shock has no effect on current output for any Δ_p. Hence, the fiscal multiplier puzzle is absent in this special case.

Finally, for the paradox of flexibility, the assumption of history dependent policy changes the relationship of price flexibility with economic outcomes in two ways. First, the dynamic multipliers all feature a κ in the denominator. Therefore, as price flexibility increases its affect on the shocks is balanced when κ also appears in the numerator. Second, price flexibility's effect through the terminal condition is finite. Taking the limit of $-\Omega^*$ as $\kappa \to \infty$ yields

$$\frac{1}{2}\left(-\sqrt{(\rho^* + \phi^*)^2} - \rho^* + \phi^*\right).$$

This resolution of the paradox of flexibility is quite intuitive. In the absence of history dependence, price stickiness is its own type of history dependence. When a central bank makes an announcement about future policy, the extent to which a price setting firm takes that information into account depends on the likelihood that they are unable to re-optimize their price when policy becomes active again. When prices are very sticky, therefore, there is a pseudo history dependence to monetary policy in this thought experiment because firms take into account the future active policy now. The less sticky prices are, the more firms are free to ignore future policy as the probability that they can reset their prices prior to the switch back to active policy is high. Explicit history dependence in policy just makes the link between current price setting and future policy responses concrete.

3.3 The full model with a sunspot

We now return to full model with $0 < \beta < 1$. The solution method used previously does not yield useful closed form solutions here. We, therefore, consider a special case that is inspired both by Cochrane (2017) and Gibbs and McClung (2023). We assume that $0 \leq \phi < 1$ but $\phi^* = 0$, while all else remains the same. We then assume that agents coordinate their expectations around the history that the central bank is tracking, ω_t. This is akin to a sunspot equilibrium and meant to be illustrative of what history dependence does in a determinate equilibrium where the central bank does indeed credibly responds to the history.

We can write the model in terms of just inflation

$$\left(\beta L^{-2} - (\beta + 1 + \frac{\kappa}{\sigma})L^{-1} + \left(1 + \frac{\kappa\phi}{\sigma}\right)\right)\pi_t = X_t. \tag{20}$$

Then, using the equation for ω_t, we express

$$\pi_t = (1 - \rho L)\omega_t.$$

We can use the above to eliminate π_t from (20) to arrive at representation of the model in terms of ω_t, the lone endogenous state variable:

$$\left(\beta L^{-2} - (\beta + 1 + \frac{\kappa}{\sigma})L^{-1} + 1\right)(1 - \rho L)\omega_t = X_t.$$

Factoring the lag polynomial, we can write the model as

$$(L^{-1} - \lambda_1)(L^{-1} - \lambda_2)(1 - \rho L)\omega_t^{\pi} = X_t. \tag{21}$$

The above expressions has three roots: $0 < \lambda_1 < 1 < \lambda_2$ and ρ. The last root is a free parameter. If we set it above one, then we have two roots outside the unit circle and one root inside the unit circle, which satisfies the Blanchard-Kahn conditions without any further assumption required to construct a unique rational expectations equilibrium. This equilibrium is similar to those constructed using the approach of Bianchi and Nicolò (2021).

The effect in time $t = T$ of a one-time anticipated shock (i^*, a^*, or g^*) that is known in T and occurring in time T^* is

$$\begin{aligned} \pi_T &= \frac{1}{\rho - \lambda_2} \sum_{k=1}^{2} \left(\lambda_2^{-\Delta_p - k} - \rho^{-\Delta_p - k} \right) X_{T^*+1-k} \\ y_T &= \frac{1-\beta}{\rho - \lambda_2} \left(\lambda_2^{-\Delta_p - 1} - \rho^{-\Delta_p - 1} \right) \frac{X_{T^*}}{\kappa} + \frac{1}{\rho - \lambda_2} \left(\lambda_2^{-\Delta_p - 2} - \rho^{-\Delta_p - 2} \right) \frac{X_{T^*-1}}{\kappa} \end{aligned}$$

Because we are solving forward with two unstable roots, we have the impact of the policies is diminished with the horizon, Δ_p. The promise to more-than-make-up for past misses provides the dampening to expectations that keeps expectations in check.

Proposition 3: *The NK model (12), (13), (16), and (17) with $\rho = \rho^* \geq 0$, $0 \leq \phi < 1$ and $\phi^* = 0$ has the following properties*

1. *The equilibrium is puzzle free if $\rho > 1$.*
2. *The magnitude of the inflation/output response to anticipated monetary or fiscal policy shocks is decreasing as ρ increases.*
3. *The equilibrium exhibits forward guidance puzzle, the fiscal multiplier puzzle, the paradox of toil, and the paradox of flexibility if $\rho = 0$.*

The example here is contrived because the central doesn't act on these beliefs. But the same dynamic carries through to the case when they do as shown in the previous example. The key point here is that the role of history dependence is made transparent. It is the lack of history dependence when policy is set passively or pegged that generates the puzzles. Once history dependence is introduced, the puzzles begin to weaken. When history dependence is large enough, the puzzles are gone.

4 Quantitative Implications

Finally, we return to the model of Smets and Wouters (2007) to investigate the quantitative implications of our finding. Specifically, we want to quantify how differently anticipated shocks transmit under an interest rate peg when the only change we make is to convert the policy rule to weighted average form when the peg ends. Figures 1 and 2 in the Introduction previewed these results. Shocks indeed transmit very differently in this scenario under the two formulations of policy.

For this exercise, we take the posterior distribution from the original Smets and Wouters (2007) paper as given. We do not re-estimate the model. We want to hold everything fixed with the exception of the form of the policy rule. We convert the monetary policy rule in the model given by

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\left(\phi_\pi \pi_t + \phi_x x_t + \phi_{dx}(x_t - x_{t-1})\right) + \epsilon_{r,t} \tag{22}$$

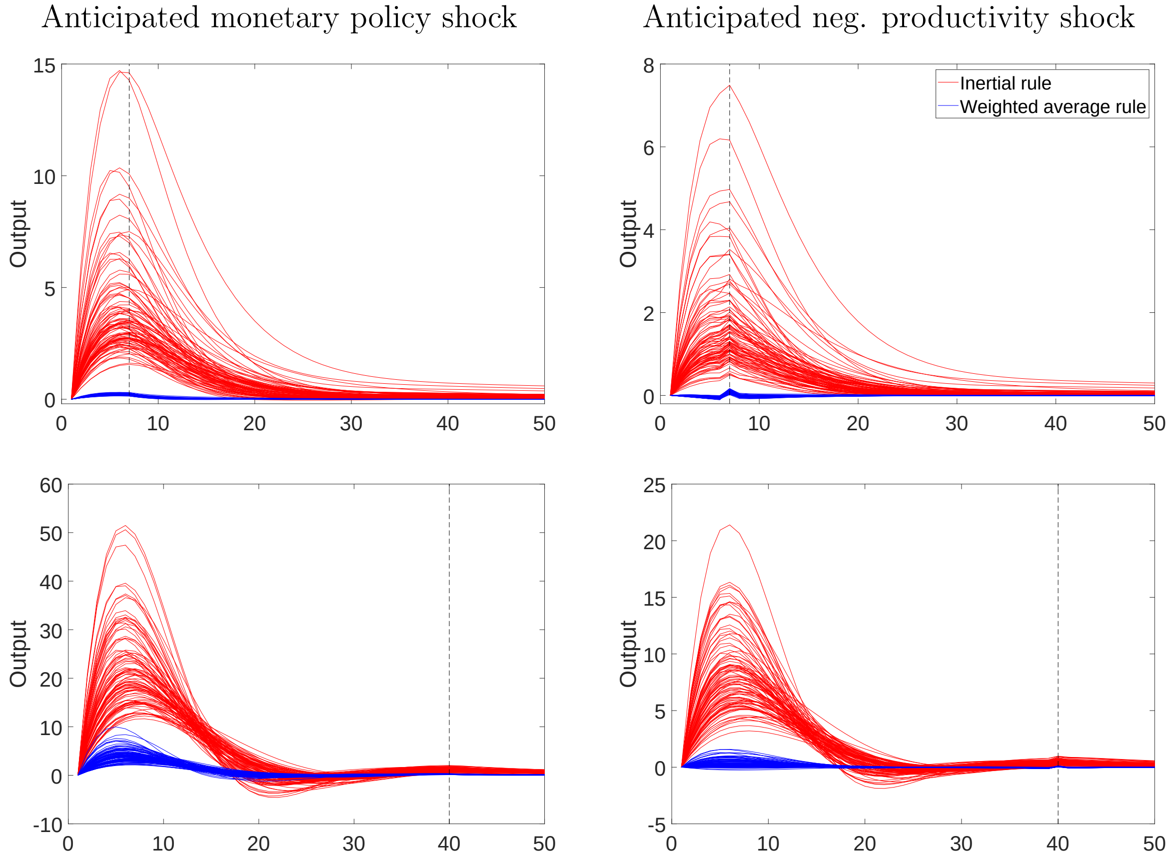
to a weighted average form given by

$$\begin{aligned} i_t &= \phi_\pi \omega_t^\pi + \phi_x \omega_t^x + \phi_{dx}\omega_t^{dx} + \omega_t^{\epsilon_r} \qquad (23)\\ \omega_t^\pi &= \omega_{t-1}^\pi + (1-\rho_i)(\pi_t - \omega_{t-1}^\pi)\\ \omega_t^x &= \omega_{t-1}^x + (1-\rho_i)(x_t - \omega_{t-1}^x)\\ \omega_t^{dx} &= \omega_{t-1}^{dx} + (1-\rho_i)(x_t - x_{t-1} - \omega_{t-1}^{dx})\\ \omega_t^{\epsilon_r} &= \omega_{t-1}^{\epsilon_r} + (1-\rho_i)(\epsilon_{r,t} - \omega_{t-1}^{\epsilon_r}) \end{aligned}$$

where x_t is the output gap. These are the same rule used to create Figures 1 and 2.

Figure 6 shows generalized impulse response functions for two shocks using draws from the model's parameter posterior distribution. The first shock is an anticipated one-off 100-basis point monetary policy shock seven (top) or forty (bottom) quarters in the future under an interest rate peg. The second shock is an anticipated one-off one standard deviation productivity shock seven (top) or forty (bottom) quarters in the future under an interest rate peg. After the shocks occur, the policy rule becomes active again and it is parameterized according to the posterior draw. The figure depicts only the responses to output for ease of comparison. The

Figure 6: An inertial rule vs a weighted average rule in the Smets and Wouters model



Notes: The black dashed line indicates when the shock occurs. Each solid line corresponds to the impulse response implied by one draw from the posterior distribution of parameters for a one-time 100-basis point monetary policy shock anticipated to occur seven or forty quarters in the future under an interest rate peg (left) or a one-time one standard deviation negative productivity shock anticipated to occur seven or forty quarters in the future under an interest rates peg (right). In period 8 (41) policy returns to either the inertial rule or the weighted average rule. Neither shock has persistence.

other endogenous variables behave in an analogous fashion. The inertial rule results shown in red display the classic forward guidance puzzle, the paradox of toil, and the paradox of flexibility. The latter result is seen by noting the significant variance in the magnitude of responses to different draws from the posterior. Much of this variation is explained by changes in wage and price stickiness. The weighted average rule does not appear to exhibit any of these puzzle.

However, the history dependence in the weighted average rule is insufficient to eliminate the puzzles since the posterior estimate of $\rho_i = 0.81$ with and 95% HPD interval of 0.77 to 0.85. This is visible by comparing the top and bottom rows of plots in Figure 6. The response of output to the anticipated shocks increase in both cases when the shock is expected to occur farther into the future. But as predicted by Propositions 1 and 2, the history dependence significantly dampens the effects of the puzzles such that they are much less relevant at business cycle frequencies.

5 Conclusion

New Keynesian models of monetary policy are known to generate puzzling predictions for output and inflation in response to anticipated shocks when nominal interest rates are pegged or constrained by the zero lower bound. We show that all New Keynesian puzzles and paradoxes are a result of assuming that monetary policy has little or no history dependence. Adding an empirically plausible amount of history dependence to monetary policy in a model can significantly mitigate the New Keynesian puzzles such that they become irrelevant for empirically plausible scenarios. Perhaps the real New Keynesian puzzle is why we entertained policy scenarios where a central bank credibly commits to ignore the variables that it is statutorily mandated to target when at the ZLB.

Appendix For Online Publication

A1 Weighted average representations

We use lag operators to convert rules with interest rate smoothing to rule that are weighted averages of past output and inflation, where $LX_t = X_{t-1}$ and $L^n X_t = X_{t-n}$ for $n = ..., -2, -1, 0, 1, 2, ...$ Following Sargent (2009), we consider polynomials in the lag operator

$$A(L) = a_0 + a_1 L + a_2 L^2 + ... = \sum_{j=0}^{\infty} a_j L^j$$

where the a_j's are constants and

$$A(L)X_t = (a_0 + a_1 L + a_2 L^2 + ...)X_t = \sum_{j=0}^{\infty} a_j X_{t-j}.$$

We exploit the following relationship

$$A(L) = \frac{1}{1 - \lambda L} = (1 + \lambda L + \lambda^2 L^2 + ...) = \sum_{j=0}^{\infty} \lambda^j,$$

where if $|\lambda| < 1$

$$\sum_{j=0}^{\infty} \lambda^j = \frac{1}{1 - \lambda}.$$

Applying these operations to a Taylor-type rule with interest rate smoothing we can derive equation (3):

$$
\begin{aligned}
i_t - \rho_i i_{t-1} &= (1-\rho_i)\bar{r} + (1-\rho_i)(\phi_\pi \pi_t + \phi_y y_t) \\
(1-\rho_i L) i_t &= (1-\rho_i)\bar{r} + (1-\rho_i)(\phi_\pi \pi_t + \phi_y y_t) \\
i_t &= \frac{1}{1-\rho_i L}\left((1-\rho_i)\bar{r} + (1-\rho_i)(\phi_\pi \pi_t + \phi_y y_t)\right) \\
&= (1-\rho_i)\sum_{j=0}^{\infty} \rho_i^j \bar{r} + (1-\rho_i)\left(\sum_{j=0}^{\infty} \rho_i^j \left(\phi_\pi \pi_{t-j} + \phi_y y_{t-j}\right)\right) \\
&= (1-\rho_i)\frac{1}{1-\rho_i}\bar{r} + (1-\rho_i)\left(\sum_{j=0}^{\infty} \rho_i^j \left(\phi_\pi \pi_{t-j} + \phi_y y_{t-j}\right)\right) \\
&= \bar{r} + (1-\rho_i)\sum_{j=0}^{\infty} \rho_i^j \left(\phi_\pi \pi_{t-j} + \phi_y y_{t-j}\right).
\end{aligned}
$$

We arrive at the representation of rule with auxiliary variables ω_t^π and ω_t^y by way of the following calculations:

$$
\begin{aligned}
\omega_t^z &= \omega_{t-1}^z + (1-\rho_i)(z_t - \omega_{t-1}^z) \\
\omega_t^z &= \rho_i \omega_{t-1}^z + (1-\rho_i) z_t \\
\omega_t^z - \rho_i \omega_{t-1}^z &= (1-\rho_i) z_t \\
(1-\rho_i L)\omega_t^z &= (1-\rho_i) z_t \\
\omega_t^z &= \frac{(1-\rho_i)}{1-\rho_i L} z_t = (1-\rho_i)\sum_{j=0}^{\infty} \rho_i^j z_{t-j} \qquad \text{(A1)}
\end{aligned}
$$

Applying this representation for $z = y, \pi$ yields equation (4).

A2 Proofs of propositions

Proposition 1: The unconditional optimal targeting criteria is

$$
y_t = -\frac{\kappa}{\alpha}\frac{\pi_t}{1-\beta L}.
$$

Using the IS equation, we can write this as

$$E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t\pi_{t+1} - r_t^n) = -\frac{\kappa}{\alpha}\frac{\pi_t}{1-\beta L}.$$

Solving for i_t we have

$$i_t = \frac{\sigma\kappa}{\alpha}\frac{\pi_t}{1-\beta L} + \sigma E_t y_{t+1} + E_t\pi_{t+1} + r_t^n.$$

The first representation of the optimal rule comes from using (A1) where

$$\begin{aligned}(1-\beta)\frac{\pi_t}{1-\beta L} &= \omega_t^\pi \\ \omega_t^\pi &= \omega_{t-1}^\pi + (1-\beta)(\pi_t - \omega_{t-1}^\pi).\end{aligned}$$

The second representation of the optimal rule comes from multiplying both sides of the equation by $(1-\beta L)$:

$$\begin{aligned}(1-\beta L)i_t &= (1-\beta L)\left(\frac{\sigma\kappa}{\alpha}\frac{\pi_t}{1-\beta L} + \sigma E_t y_{t+1} + E_t\pi_{t+1} + r_t^n\right) \\ i_t - \beta i_{t-1} &= \frac{\sigma\kappa}{\alpha}\pi_t + (1-\beta L)\left(\sigma E_t y_{t+1} + E_t\pi_{t+1} + r_t^n\right).\end{aligned}$$

Proposition 2: Consider first the calculation of the perfect foresight solution in the terminal regime. We write the reduced form model in matrix form as

$$\begin{pmatrix} 1 & \frac{\kappa\phi^*}{\sigma} \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \pi_t \\ \omega_t^\pi \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \rho^* \end{pmatrix}\begin{pmatrix} \pi_{t-1} \\ \omega_{t-1}^\pi \end{pmatrix} + \begin{pmatrix} \frac{\sigma+\kappa}{\sigma} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E_t\pi_{t+1} \\ E_t\omega_{t+1}^\pi \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} X_t$$

Rearranging matrices we have

$$\begin{pmatrix} \pi_t \\ \omega_t^\pi \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\kappa\rho^*\phi^*}{\sigma+\kappa\phi^*} \\ 0 & \frac{\rho^*\sigma}{\sigma+\kappa\phi^*} \end{pmatrix}\begin{pmatrix} \pi_{t-1} \\ \omega_{t-1}^\pi \end{pmatrix} + \begin{pmatrix} \frac{\kappa+\sigma}{\sigma+\kappa\phi^*} & 0 \\ \frac{\kappa+\sigma}{\sigma+\kappa\phi^*} & 0 \end{pmatrix}\begin{pmatrix} E_t\pi_{t+1} \\ E_t\omega_{t+1}^\pi \end{pmatrix} + \begin{pmatrix} \frac{\sigma}{\sigma+\kappa\phi^*} \\ \frac{\sigma}{\sigma+\kappa\phi^*} \end{pmatrix} X_t$$

Using the method of undetermined coefficients, the solution is

$$\begin{pmatrix} \pi_t \\ \omega_t^{\pi} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\rho^*\sigma-\sigma+\sqrt{(\sigma(\rho^*-1)+\kappa(\rho^*-\phi^*))^2+4\kappa(\kappa+\sigma)\rho^*\phi^*}+\kappa\rho^*-\kappa\phi^*}{2(\kappa+\sigma)} \\ 0 & \frac{\rho^*\sigma+\sigma-\sqrt{(\sigma(\rho^*-1)+\kappa(\rho^*-\phi^*))^2+4\kappa(\kappa+\sigma)\rho^*\phi^*}+\kappa\rho^*+\kappa\phi^*}{2(\kappa+\sigma)} \end{pmatrix} \begin{pmatrix} \pi_{t-1} \\ \omega_{t-1}^{\pi} \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{2\sigma}{\rho^*\sigma+\sigma+\sqrt{(\sigma(\rho^*-1)+\kappa(\rho^*-\phi^*))^2+4\kappa(\kappa+\sigma)\rho^*\phi^*}+\kappa\rho^*+\kappa\phi^*} \\ \frac{2\sigma}{\rho^*\sigma+\sigma+\sqrt{(\sigma(\rho^*-1)+\kappa(\rho^*-\phi^*))^2+4\kappa(\kappa+\sigma)\rho^*\phi^*}+\kappa\rho^*+\kappa\phi^*} \end{pmatrix} X_t$$

Therefore,

$$\begin{aligned} \Omega^* &:= \frac{\rho^*\sigma-\sigma+\sqrt{\left(\sigma\left(\rho^*-1\right)+\kappa\left(\rho^*-\phi^*\right)\right)^2+4\kappa(\kappa+\sigma)\rho^*\phi^*}+\kappa\rho^*-\kappa\phi^*}{2(\kappa+\sigma)} > 0 \\ C^* &:= \frac{2\sigma}{\rho^*\sigma+\sigma+\sqrt{\left(\sigma\left(\rho^*-1\right)+\kappa\left(\rho^*-\phi^*\right)\right)^2+4\kappa(\kappa+\sigma)\rho^*\phi^*}+\kappa\rho^*+\kappa\phi^*} \end{aligned}$$

The puzzle resolutions follow from the properties of $\Phi(\Delta_p)$. By assumption $0 < \psi < 1$. We can rewrite this expression as

$$\Phi(\Delta_p) = \frac{\Omega^*\left(\rho^{\Delta_p+1}-\psi^{\Delta_p+1}\right)}{\rho-\psi} + \psi^{\Delta_p+1}$$

The following limits are easily obtained

1. when $0 < \rho < 1$ then
$$\lim_{\Delta_p\to\infty} \Phi(\Delta_p) = 0 \tag{A2}$$

2. when $\rho = 1$, then
$$\lim_{\Delta_p\to\infty} \Phi(\Delta_p) = (1-\psi)^{-1}\Omega^* \tag{A3}$$

3. when $\rho > 1$, then
$$\lim_{\Delta_p\to\infty} \Phi(\Delta_p) = +\infty \tag{A4}$$

4. when $\rho \geq 0$ and $\phi > 0$, then
$$\lim_{\kappa\to\infty} \Phi(\Delta_p) = \frac{\left(\sqrt{\left(\rho^*+\phi^*\right)^2}+\rho^*-\phi^*\right)\left(\rho^{\Delta_p+1}-\phi^{\Delta_p+1}\right)}{2(\rho-\phi)} + \phi^{\Delta_p+1} \tag{A5}$$

5. when $\rho = 0$ and $\phi = 0$, then
$$\lim_{\kappa\to\infty} \Phi(\Delta_p) = 0 \tag{A6}$$

Proposition 2 part 3 states that the puzzles are decreasing as ρ is increasing. This requires that $\partial\Phi(\Delta_p)/\partial\rho > 0$.

$$\frac{\partial\Phi(\Delta_p)}{\partial\rho} = \frac{\Omega^* \left(\psi\left(\psi^{\Delta_p} - \rho^{\Delta_p}\right) + \Delta_p(\rho - \psi)\rho^{\Delta_p}\right)}{(\rho-\psi)^2}$$

To see this always positive note that when $\rho = 0$, the above is unambiguously positive.

The inflation and output response to an anticipated shock are given by

$$\begin{aligned}\pi_T &= \Phi(\Delta_p)^{-1}\frac{X_{T^*} + (\psi+\Omega^*)X_{T^*-1}}{1+\kappa\sigma^{-1}}\\ y_T &= \frac{1}{\kappa}\Phi(\Delta_p)^{-1}\frac{X_{T^*} + (\psi+\Omega^*)X_{T^*-1}}{1+\kappa\sigma^{-1}}.\end{aligned}$$

Interest rate shock: The dynamic multiplier for an interest rate shock is:

$$\partial y_t/\partial i_{t+\Delta_p} = -\frac{1}{\kappa}\Phi(\Delta_p)^{-1}\frac{\kappa\sigma^{-1}}{1+\kappa\sigma^{-1}},\ \partial\pi_t/\partial i_{t+\Delta_p} = -\Phi(\Delta_p)^{-1}\frac{\kappa\sigma^{-1}}{1+\kappa\sigma^{-1}}.$$

When $0 \leq \rho < 1$, then the limit of $\Phi(\Delta_p)$ is given by (A2) and the forward guidance puzzle is present. When $\rho = 1$, the limit of $\Phi(\Delta_p)$ is given by (A3) and forward guidance puzzle is absent. However, there is a weak anti-horizon effect, where the effect of forward guidance doesn't go to zero even when the anticipated shock is infinitely far in the future. When $\rho > 1$, then the limit of $\Phi(\Delta_p)$ is given by (A4) and the forward guidance puzzle is eliminated. The paradox of flexibility only occurs for this shock when $\rho = 0$.

Government spending shock: The dynamic multiplier for a government spending shock is:

$$\partial y_t/\partial g_{t+\Delta_p} = (1-\delta_g)\Phi(\Delta_p)^{-1}\frac{(1-\psi-\Omega^*)}{1+\kappa\sigma^{-1}},\ \partial\pi_t/\partial g_{t+\Delta_p} = \kappa(1-\delta_g)\Phi(\Delta_p)^{-1}\frac{(1-\psi-\Omega^*)}{1+\kappa\sigma^{-1}}.$$

When $0 \leq \rho < 1$, then the limit of $\Phi(\Delta_p)$ is given by (A2) and the forward guidance puzzle is present, except in the special case where

$$\rho^* = \bar{\rho} := \frac{(1-\phi)\left(\kappa\phi^* + \kappa(1-\phi) + \sigma\right)}{(\phi^* - \phi + 1)(\kappa + \sigma)}.$$

where by assumption $0 \leq \phi < 1$ and $\phi^* > 1$. If $\rho^* = \bar{\rho}$ then $\psi + \Omega^* - 1 = 0$ and therefore inflation and output do not respond to the anticipated fiscal shock. The fiscal forward guidance puzzle is eliminated when $\rho \geq 1$. The paradox of flexibility is only present when $\rho = 0$ and $\phi = 0$.

Supply shock: The dynamic multiplier for a supply shock is:

$$\partial y_t / \partial a_{t+\Delta_p} = \Phi(\Delta_p)^{-1} \frac{(\psi + \Omega^* - 1)}{1 + \kappa\sigma^{-1}}, \; \partial \pi_t / \partial a_{t+\Delta_p} = \kappa\Phi(\Delta_p)^{-1} \frac{(\psi + \Omega^* - 1)}{1 + \kappa\sigma^{-1}}.$$

This take the same form as the government spending shock. The effect of this shock goes to zero as $\Delta_p \to \infty$ when $\rho > 1$ by (A4).

Resolving the paradox of toil requires

$$\partial y_t / \partial a_{t+\Delta_p} = \Phi(\Delta_p)^{-1} \frac{(\psi + \Omega^* - 1)}{1 + \kappa\sigma^{-1}} \geq 0.$$

This expression is positive so long as

$$\psi + \Omega^* > 1.$$

which holds if and only if $\rho^* > \bar{\rho}$.

Proposition 3: We first derive the solution for inflation by solving forward with λ_2 and ρ.

$$\begin{aligned} (L^{-1} - \lambda_1)(L^{-1} - \lambda_2)(1 - \rho L)\omega_t^{\pi} &= X_t \\ (L^{-1} - \lambda_1)(\rho\omega_{t-1} + \pi_t) &= \frac{1}{(L^{-1} - \lambda_2)(1 - \rho L)} X_t \end{aligned}$$

We split the right hand side using partial fractions

$$\frac{1}{(L^{-1}-\lambda_2)(1-\rho L)} = \frac{A}{(L^{-1}-\lambda_2)} + \frac{B}{(1-\rho L)}$$

where when $L = \lambda_2^{-1}$ we have

$$\begin{aligned}
1 &= A(1-\rho L) + B(L^{-1}-\lambda_2) \\
&= A(1-\rho\lambda_2^{-1}) + B(\lambda_2-\lambda_2) \\
&= A\left(\frac{\lambda_2-\rho}{\lambda_2}\right) \\
&\dots \\
A &= \frac{\lambda_2}{\lambda_2-\rho}
\end{aligned}$$

and when $L = \rho^{-1}$ we have

$$\begin{aligned}
1 &= A(1-\rho L) + B(L^{-1}-\lambda_2) \\
&= A(1-\rho\rho^{-1}) + B(\rho-\lambda_2) \\
&= B(\rho-\lambda_2) \\
&\dots \\
B &= \frac{1}{\rho-\lambda_2}
\end{aligned}$$

Then we can write

$$\begin{aligned}
(L^{-1}-\lambda_1)(\rho\omega_{t-1}+\pi_t) &= \frac{1}{\rho-\lambda_2}\left(\frac{-\lambda_2}{(L^{-1}-\lambda_2)} + \frac{1}{(1-\rho L)}\right)X_t \\
&= \frac{1}{\rho-\lambda_2}\left(\frac{-\lambda_2}{-\lambda_2(1-(\lambda_2 L)^{-1})} + \frac{1}{-(\rho L)(1-(\rho L)^{-1})}\right)X_t \\
&= \frac{1}{\rho-\lambda_2}\left(\frac{1}{(1-(\lambda_2 L)^{-1})} - \frac{(\rho L)^{-1}}{(1-(\rho L)^{-1})}\right)X_t.
\end{aligned}$$

We assume that for $t < T$ the economy is at steady state such that $\pi_{T-j} = \omega_{T-j} = 0$ for $j = 1, 2, \ldots$. With this assumption we can write the above as

$$\begin{aligned}
\rho\omega_{T-1} + \pi_T - \lambda_1(\rho\omega_{T-2} + \pi_{T-1}) &= \frac{1}{\rho - \lambda_2}\left(\frac{1}{(1-(\lambda_2 L)^{-1})} - \frac{(\rho L)^{-1}}{(1-(\rho L)^{-1})}\right) X_{T-1} \\
\pi_T &= \frac{1}{\rho - \lambda_2}\left(\frac{1}{(1-(\lambda_2 L)^{-1})} - \frac{(\rho L)^{-1}}{(1-(\rho L)^{-1})}\right) X_{T-1}
\end{aligned}$$

Imposing the policy (16) and (17) and assuming that shocks are non-zero only in $t = T^*$ such that $X_j = 0$ except when $t = T^*$ or $t = T^* - 1$

$$\begin{aligned}
\pi_T &= \frac{1}{\rho - \lambda_2}\left(\sum_{j=0}^{T^*+1}\left(\frac{1}{\lambda_2}\right)^j X_{T+j-1} - \frac{1}{\rho}\sum_{j=0}^{T^*}\left(\frac{1}{\rho}\right)^j X_{T+j}\right) \\
&= \frac{1}{\rho - \lambda_2}\sum_{k=1}^{2}\left(\lambda_2^{-\Delta_p - k} - \rho^{-\Delta_p - k}\right) X_{T^*+1-k} \\
&= \frac{1}{\rho - \lambda_2}\left(\lambda_2^{-\Delta_p - 1} - \rho^{-\Delta_p - 1}\right)\frac{\kappa}{\sigma}(\bar{r} - i^*) \\
&\quad + \left(\frac{1}{\rho - \lambda_2}\left(\lambda_2^{-\Delta_p - 1} - \rho^{-\Delta_p - 1}\right) - \frac{1}{\rho - \lambda_2}\left(\lambda_2^{-\Delta_p - 2} - \rho^{-\Delta_p - 2}\right)\right)\kappa(1 - \delta_g)g^* \\
&\quad - \left(\frac{1}{\rho - \lambda_2}\left(\lambda_2^{-\Delta_p - 1} - \rho^{-\Delta_p - 1}\right) - \frac{1}{\rho - \lambda_2}\left(\lambda_2^{-\Delta_p - 2} - \rho^{-\Delta_p - 2}\right)\right)\kappa a^* \\
&\ldots \\
\pi_T &= \frac{1}{\rho - \lambda_2}\left\{\left(\lambda_2^{-\Delta_p - 1} - \rho^{-\Delta_p - 1}\right)\frac{\kappa}{\sigma}(\bar{r} - i^*) + \left((\lambda_2 - 1)\lambda_2^{-\Delta_p - 2} + (1 - \rho)\rho^{-\Delta_p - 2}\right)\kappa\left((1 - \delta_g)g^* - a^*\right)\right\}
\end{aligned}$$

To solve for the equilibrium output response, we start by rearranging the Phillips curve

$$\kappa y_{t+1} = \pi_t - \beta\pi_{t+1}.$$

Starting from period T, we have

$$\begin{aligned}
\kappa y_T &= \frac{1}{\rho - \lambda_2}\sum_{k=1}^{2}\left(\lambda_2^{-\Delta_p - k} - \rho^{-\Delta_p - k}\right) X_{T^*+1-k} - \beta\frac{1}{\rho - \lambda_2}\left(\lambda_2^{-\Delta_p - k} - \rho^{-\Delta_p - k}\right) X_{T^*} \\
&= \frac{1 - \beta}{\rho - \lambda_2}\left(\lambda_2^{-\Delta_p - 1} - \rho^{-\Delta_p - 1}\right) X_{T^*} + \frac{1}{\rho - \lambda_2}\left(\lambda_2^{-\Delta_p - 2} - \rho^{-\Delta_p - 2}\right) X_{T^*-1} \\
&\ldots \\
y_T &= \frac{1 - \beta}{\rho - \lambda_2}\left(\lambda_2^{-\Delta_p - 1} - \rho^{-\Delta_p - 1}\right)\frac{1}{\sigma}(\bar{r} - i^*) \\
&\quad + \frac{1}{\rho - \lambda_2}\left\{\left(((1 - \beta)\lambda_2 - 1)\lambda_2^{-\Delta_p - 2} + (1 - (1 - \beta)\rho)\rho^{-\Delta_p - 2}\right)\left((1 - \delta_g)g^* - a^*\right)\right\}
\end{aligned}$$

For part 1 of the proposition, we note that $\lambda_2 > 1$ and $\rho > 1$ by assumption. When we send $\Delta_p \to \infty$, the expressions for π_T and y_T converge to zero. Therefore, the forward guidance puzzle and the fiscal multiplier puzzle are not present. For the paradox of toil, we have

$$y_T = \frac{1}{\rho - \lambda_2}\left\{\left(((1-\beta)\lambda_2 - 1)\lambda_2^{-\Delta_p - 2} + (1 - (1-\beta)\rho)\rho^{-\Delta_p - 2}\right)(-a^*)\right\}$$

Consider the case when $\rho = 1$, then

$$\frac{\partial y_T}{\partial a^*} = \frac{\left(\lambda_2 - \beta\lambda_2 + \beta\lambda_2^{\Delta_p+2} - 1\right)}{\lambda_2 - 1}\lambda_2^{-\Delta_p-2} > 0$$

because $\lambda_2 > 1$. Consider the case when $\rho \to \lambda_2$ from either above or below

$$\lim_{\rho \to \lambda_2} \frac{\partial y_T}{\partial a^*} = \lambda_2^{-\Delta_p - 3}\left(2 + \Delta_p - (1-\beta)\lambda_2(1 + \Delta_p)\right)$$

A typical value for $\beta = 0.99$. Typical parameterizations of the model imply λ_2 close to one, which means that second term in the parentheses is small in absolute value relative to the first, which means the multiplier is positive. Finally, taking the limit as $\rho \to 0$ of $\frac{\partial y_T}{\partial a^*}$ is zero. Therefore, we conclude that the paradox of toil is absent for all $\rho \geq 1$.

For the paradox of flexibility, we look at λ_2, which is equal to

$$\lambda_2 = \frac{\sqrt{\left(-\beta - \frac{\kappa}{\sigma} - 1\right)^2 - 4\beta\left(\frac{\kappa\phi}{\sigma} + 1\right)} + \beta + \frac{\kappa}{\sigma} + 1}{2\beta}$$

and

$$\lim_{\kappa \to \infty} \lambda_2 = \infty.$$

In addition,

$$\lim_{\kappa \to \infty} \lambda_2^{-1}\kappa = \frac{2\beta\sigma}{\sigma\sqrt{\frac{1}{\sigma^2} + 1}}.$$

so the paradox of flexibility is absent.

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